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Limit cycles in sampled-data relay feedback systems

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Abstract Analysis of limit-cycles in relay feedback systems is usually performed in continuous time, even though most such systems are implemented digitally. In this paper we discuss the limitations of the continuous time analysis, showing that even for standard plants with reasonable sampling rates its results can be considerably far from the truth. Then we present a discrete time analysis of relay feedback systems, providing analytical tools that overcome the limitations of their continuous time counterparts.

1 Introduction

Analysis of relay feedback systems is a classical topic in control systems theory, and prediction of the modes of self-oscillation - that is, the limit-cycles - is the central issue in this analysis [17][19]. Early work on analysis of limit-cycles in relay feedback systems in the context of control theory dates back to the 1950's [6][12][15]. From the works of that time, the continuous-time describing function analysis emerged as the standard tool for analysis of limit-cycles in nonlinear systems, due to its simplicity and graphical appeal.

New interest in relay feedback arose in the 1980's motivated by the idea of using relays for tuning PI and PID controllers [1][9]. A whole family of tuning methods have been developed, whose application has become widespread in industrial practice for the tuning of single-loop controllers, and similar methods for the tuning of multivariable controllers have also been proposed [5][8]. This created a need for better understanding of the behaviour of relay feedback systems, and a more precise description of the limit-cycles than the one provided by the approximate describing function analysis. State space analysis has provided these more precise analytical tools for the prediction of limit-cycles and also a more complete understanding of other phenomena occurring in relay feedback systems [11][18]. The interest in developing new tuning methods based on relay experiments continues to this day - see for instance the tuning rules for resonant controllers presented in [16] and the ones for event-based controllers in [4]. All these tuning methods (the now classical methods from the 1980's as well as the newly developed) rely on a correct theoretical prediction of the limit-cycles to be observed in an actual relay feedback experiment for the derivation of their tuning formulas. Accordingly, analytical tools for better describing the relay feedback systems continue to appear [10][7].

Even though the relay feedback systems are mostly implemented digitally, their analysis is usually performed in continuous time, neglecting the sampled-data characteristic of the system. This analysis relies on the sampling rates being fast enough for its results to be correct. Several of the early works on relay feedback systems did not ignore the sampling issue in the analysis, either dealing explicitly with the sampled-data nature of the implementation [6] or even performing the analysis strictly in discrete-time [15]. Yet, continuous-time analysis has become the standard in nonlinear systems in general, relay feedback systems not being an exception, and these early discrete-time results did not seem to have developed into a complete and contemporary theory, contrary to the continuous-time analysis. We will show by means of a quite standard case study that the period and amplitude of oscillation in a sampled-data relay feedback system can be significantly distant from the ones predicted by the continuous time analysis even for seemingly appropriate sampling rates. Tuning a controller based on the information collected from such an experiment could thus result in poor tuning. In this paper, sampled relay feedback systems are analysed in the discrete time domain in order to obtain more precise descriptions of their limit-cycles.

The paper is organized as follows. The formal definition of the problem under study is given in Section 2 and continuous time analysis tools are presented in Section 3. These tools are applied to a standard case

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study, and it is shown in Section 4 that the periods and amplitudes observed in the actual sampled data system can be significantly far from the ones predicted, even for sampling rates within the range that control textbooks recommend for this plant. Then we develop, in Section 5, the discrete-time analysis of limit-cycles. Two exact methods are derived, one in the time domain, using the state space representation of the plant, and another one in the frequency domain. An approximate method is also derived from the frequency domain approach. It is shown by comparison of these analytical tools with their continuous-time counterparts that more limit-cycles are to be expected in the sampled-data system than in the original continuous-time system. These tools are then applied to the case study, showing that they predict all limit-cycles correctly, overcoming the limitations of the continuous-time analysis.

2 Preliminaries

2.1 Definitions and Notation

We consider linear time-invariant SISO systems in continuous time, which can be described by an input-output relationship

$$\frac{Y(s)}{U(s)} = G(s) \tag{1}$$

where $Y(s)$, $U(s)$ and $G(s)$ are the Laplace transforms of the system's output, input and impulse response, respectively. It is assumed that the transfer function $G(s)$ is rational and strictly proper.[1] We will also consider minimal state space realizations of (1):

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bu(t) \tag{2}$$

$$y(t) = C\mathbf{x}(t) \tag{3}$$

with $G(s) = C(sI - A)^{-1}B$, (A, B) controllable and (C, A) observable.

When the input and output of the continuous-time system (1) are sampled with a zero-order-holder, the resulting discrete-time system can be described by the input-output relationship:

$$\frac{Y(z)}{U(z)} = G(z) \tag{4}$$

where $Y(z)$ and $U(z)$ are the Z transforms of the system's output and input, respectively, and $G(z) = (1 - z^{-1})\mathcal{Z}\{\mathcal{L}^{-1}[\frac{G(s)}{s}]\}$. A minimal realization of the discrete-time system (4) can be written as:

$$\mathbf{x}(k+1) = \Phi\mathbf{x}(k) + \Psi u(k) \tag{5}$$

$$y(k) = C\mathbf{x}(k) \tag{6}$$

[1] So that its feedback connection with a static element is well-posed.

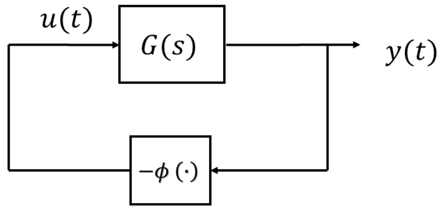


Fig. 1 Feedback connection of continuous time plant and nonlinear element

with $G(z) = C(zI - \Phi)^{-1}\Psi$. We assume that the sampling rate is such that the controllability and observability properties of the continuous time system are not lost, thus (Φ, Ψ) is controllable and (C, Φ) is observable. Moreover, the realizations (5) and (2) are related by:

$$\Phi = e^{AT_s} \tag{7}$$

$$\Psi = \int_0^{T_s} e^{At} dt\, B \tag{8}$$

where T_s is the sampling period. For convenience of notation, we will write (8) as

$$\Psi = A^{-1}(\Phi - I)B \tag{9}$$

but it must be noted that the discrete-time input matrix Ψ is well-defined even if A does not have an inverse [18].

2.2 The feedback connection

We consider feedback nonlinear systems in the form depicted in Figure 1. This feedback system consists of a linear time-invariant system connected in feedback with a time-invariant and memoryless nonlinear element. Since the transfer function of the linear system has been assumed to be strictly proper, the feedback connection is well-posed. The linear system is described by (1) and (2), whereas the nonlinear element is described by

$$u = -\phi(y) \tag{10}$$

where $\phi(\cdot) : \mathbb{R} \to \mathbb{R}$ is an odd function, that is, $\phi(y) = -\phi(-y)\ \forall y \in \mathbb{R}$.

This is the standard setting in the study of existence and characterisation of limit-cycles in nonlinear systems, and ample theory on the subject is available - see [13], for instance. Before we proceed, let us formalize the subject of our study: the limit-cycle.

Definition 1 A **limit-cycle** is an isolated periodic orbit γ of an autonomous nonlinear system. If for all $\mathbf{x} \in \gamma$ the reciprocal state $-\mathbf{x}$ also belongs to γ, then the limit-cycle is said to be *symmetric*. A limit-cycle is said to be *unimodal* if the linear system's output $y(t)$ changes signal exactly twice per period. For a symmetric and unimodal limit-cycle the acronym *SULC* will be used.

◇

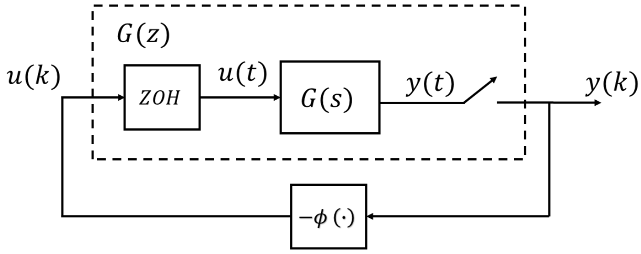


Fig. 2 Feedback connection of discrete-time (sampled continuous time) plant and nonlinear element

2.3 Relay feedback and the ultimate point

A very important particular case of the feedback element $\phi(\cdot)$ is the so-called relay feedback:

$$u(t) = -sign(y(t)). \tag{11}$$

where $sign(\cdot)$ is the sign function ($sign(x) = 1$ for positive x and $sign(x) = -1$ for negative x). Relay feedback with different amplitudes, that is $u(t) = -d.sign(y(t))$ with $d \in \mathbb{R}^+$, can be easily accommodated by multiplying the transfer function by d. Accordingly, we will assume $d = 1$ in our analysis, with no loss of generality.

Relay feedback systems arise in a wide variety of engineering applications, like failure diagnosis and controller tuning, for instance, as well as in nature, which motivates their study in biology. In the control systems framework, relay feedback experiments are mostly used as a means to identify the ultimate quantities of an unknown plant, an information that feeds tuning formulas por PID (Proportional-Integral-Derivative) and PR (Proportional-Resonant) controllers. The ultimate quantities are the characteristics of the so-called ultimate point of the frequency response of a linear system, which is the point at which the frequency response's phase reaches $-\pi$. These are the ultimate frequency ω_u and the ultimate gain K_u, which are defined as

$$\omega_u = \min_{\omega \geq 0} \omega : \angle G(\jmath\omega) = -\pi$$
$$K_u = \frac{1}{\mid G(\jmath\omega_u) \mid}.$$

Once a symmetric oscillation is obtained in the relay experiment, its amplitude A_u and period T_u are measured and the ultimate quantities are calculated from [1]

$$K_u = \frac{4}{\pi A_u} \quad \omega_u = \frac{2\pi}{T_u} \tag{12}$$

The standard tool for analysis of limit-cycles is the describing function method, which is ubiquitous in control systems textbooks, even at the undergraduate level. The describing function method provides an approximate solution to the problem. In relay feedback systems it is also possible to perform an exact analysis in the time-domain, which will be reviewed in Section 3. We will also present an alternative exact method for the analysis of limit-cycles that is based in the frequency-domain, of which the describing function result for relay feedback systems can be seen as an approximation.

2.4 Sampled-data relay feedback

The purpose of this paper is to investigate the behaviour of the sampled version of this feedback connection, that is, systems in the form depicted in Figure 2. In this system, the input and output of the linear continuous time plant have been sampled with a sampling rate T_s, and the feedback connection is otherwise unaltered with respect to the feedback connection described in Figure 1. Clearly, for fast enough sampling rates the continuous time theory should apply, but we will show by means of a simple example that the effect of sampling can be quite significant and not only quantitative but also qualitative in nature, even for seemingly appropriate sampling rates. Then we will develop analysis tools in the discrete time domain that will allow the removal of these limitations of the continuous time analysis and provide more precise descriptions of the limit cycles in these sampled data relay feedback systems. But first let us review the known tools for analysis of the continuous time system in Figure 1 and present an alternative one that will also prove itself useful.

3 Continuous time analysis

In this Section the behaviour of the continuous-time feedback connection (1)(11) (equivalently, (2)(3)(11)) is analyzed. Let us define the switching surface $\mathcal{S}$ as

$$\mathcal{S} = \{\mathbf{x} : C\mathbf{x} = 0\}. \tag{13}$$

Whenever the state of the system crosses the switching surface $\mathcal{S}$, the relay output changes signal and we say that a switching has occurred. The switching plane divides the state space into two subspaces $\mathcal{X}_+$ and $\mathcal{X}_-$:

$$\mathcal{X}_+ = \{\mathbf{x} : C\mathbf{x} > 0\} \tag{14}$$
$$\mathcal{X}_- = \{\mathbf{x} : C\mathbf{x} < 0\} \tag{15}$$

At either subspace the system behaves linearly, that is, it obeys the following equations:

$$\dot{\mathbf{x}} = A\mathbf{x} - B \quad \forall \mathbf{x} \in \mathcal{X}_+ \tag{16}$$
$$\dot{\mathbf{x}} = A\mathbf{x} + B \quad \forall \mathbf{x} \in \mathcal{X}_- \tag{17}$$

3.1 Equilibria

Within the subspace $\mathcal{X}_+$ the state evolution obeys equation (16), which presents an equilibrium point at $\mathbf{x}_+^e = A^{-1}B$. If A is Hurwitz, then any trajectory starting inside $\mathcal{X}_+$ will tend toward $\mathbf{x}_+^e$. Assume that $\mathbf{x}_+^e$ belongs to $\mathcal{X}_-$; this implies that a switching will occur, after which the system behaves according to equation (17). But the equilibrium of this last equation is $-\mathbf{x}_+^e \in \mathcal{X}_+$, thus another switching will occur as the trajectory tends to this new equilibrium, and so on. Hence, $\mathbf{x}_+^e = A^{-1}B \in \mathcal{X}_-$ is a sufficient condition for the trajectories of the relay feedback system to switch indefinitely. Now note that $C\mathbf{x}_+^e = CA^{-1}B = -G(0)$ and thus $\mathbf{x}_+^e \in \mathcal{X}_-$ can be read as $G(0) > 0$. Therefore, positive static gain is a sufficient condition for a BIBO system with relay feedback (11) to present an infinite number of switches and never reach an equilibrium, as formally stated in the following theorem [18].

Theorem 1 *If $G(0) > 0$ then the nonlinear system (1)(11) does not have equilibria and any trajectory starting away from $\mathbf{x} = \mathbf{0}$ results in an infinite number of switches.*

◇

Note that this does not imply that there is a limit-cycle, since nothing has been said about these switches being periodical. Analysis of limit-cycles starts in the following Subsection.

3.2 Time domain analysis of SULCs

The material in this Subsection summarizes the results of [18]. Assume that there exists a SULC with half-period h. Let $\mathbf{x}^*$ and $\mathbf{x}^{**}$ be the two switching points. Then

$$\mathbf{x}^{**} = e^{Ah}\mathbf{x}^* + \int_0^h e^{A(t-h)}dtB \qquad (18)$$

$$\mathbf{x}^* = e^{Ah}\mathbf{x}^{**} - \int_0^h e^{A(t-h)}dtB \qquad (19)$$

Substituting (18) into (19) gives[2]

$$\mathbf{x}^* = (I + e^{Ah})^{-1}A^{-1}(e^{Ah} - I)B \qquad (20)$$

Because the assumed limit-cycle is symmetric, $\mathbf{x}^* = -\mathbf{x}^{**}$, that is, the oscillation must be symmetric around the origin. We shall call $\mathbf{x}^*$ given by (20) the switching point. The switching point must belong to the Kernel of the output matrix, that is

$$C\mathbf{x}^* = C(I + e^{Ah})^{-1}A^{-1}(e^{Ah} - I)B = 0 \qquad (21)$$

Solving equation (21) for h gives the *exact* values of the half-periods of possible oscillations. A solution h^* of (21) is the half-period of a SULC IFF we can guarantee that there are no other switchings within this half-period, that is, IFF

$$y(t)\mid_{\mathbf{x}(0)=\mathbf{x}^*} > 0 \;\forall t \in (0,\; h^*) \qquad (22)$$

We formalize these results in a theorem [11][18].

Theorem 2 *The relay feedback system defined in (1)(11) presents a SULC with period $T = 2h^*$ if and only if h^* satisfies equations (21) and (22).*

◇

The existence of a SULC can be established by solving numerically equation (21) and then verifying, through the simulation of the system, if (22) is satisfied.

3.3 Frequency response analysis of SULCs

Let us now derive new frequency domain conditions for the existence of SULCs. If a symmetric unimodal limit-cycle with period $T = 2h$ is observed in a relay feedback system, then the linear system's input is a square wave with period T. Taking as the time reference an instant at which the square wave switches up - that is, from -1 to $+1$ - then this square wave is described by

$$\begin{aligned} u(t) &= +1 \qquad t \in (mT, h + mT) \\ u(t) &= -1 \qquad t \in (h + mT, T + mT) \end{aligned} \qquad (23)$$

for all $m \in \mathbb{Z}$. The expansion of the square wave (23) in Fourier series yields:

$$u(t) = \sum_{n=-\infty}^{\infty} c_n e^{\jmath n\omega_0 t} \qquad (24)$$

where $\omega_0 = \frac{2\pi}{T}$ is the fundamental frequency and the coefficients of the Fourier series are given by:[14]

$$c_n = -\jmath\frac{2}{\pi n} \quad n \text{ odd} \qquad c_n = 0 \quad n \text{ even}. \qquad (25)$$

The input of the plant is the output of the relay, which switches up when its input changes signal from positive to negative. But the relay's input is the plant's output, so $u(t)$ will be given by (23) if and only if the

[2] Again, for convenience of notation we will write $\int_0^h e^{A(t-h)}dt = -A^{-1}(e^{Ah} - I)$, which does not imply assuming invertibility of A

plant's output changes signal at the same time instants and in the opposite direction, that is,

$y(0) = 0$
$y(k) < 0 \; t \in (0, h)$
$y(k) > 0 \; t \in (h, T)$
$y(k) < 0 \; t \in (T, T+h)$

and so on. The output of the linear plant must satisfy these equations, crossing zero at exactly the time instants $t = 0, \pi/\omega_0, 2\pi/\omega_0, \ldots$, for the square wave (23) to exist in the feedback loop. But if the input of the plant is the square wave (23) then its output is given by:

$$y(t) = \sum_{n=-\infty}^{\infty} a_n e^{\jmath n\omega_0 t} \tag{26}$$

where

$$a_n = G(\jmath n\omega_0)c_n = \jmath\frac{4}{\pi n}G(\jmath n\omega_0) \tag{27}$$

Evaluating (26) at $t = 0$ gives

$$y(0) = \sum_{n=-\infty}^{\infty} a_n = a_0 + 2\sum_{n=1}^{\infty} Re(a_n) \tag{28}$$

where $Re(\cdot)$ indicates the real part of a complex number and the last equality comes from the fact that $a_n = a_n^*$ - the Fourier series symmetry for real signals. On the other hand, $a_0 = 0$ because $c_0 = 0$; putting this and (27) into (28) gives

$$y(0) = 2\sum_{n=1}^{\infty}\frac{4}{\pi n}Re(\jmath G(\jmath n\omega_0)) = 0$$

or just

$$\sum_{n=1}^{\infty}\frac{1}{n}Im(G(\jmath n\omega_0)) = 0 \tag{29}$$

where $Im(\cdot)$ stands for the imaginary part of a complex number. This is an equation that the fundamental frequency ω_0 must satisfy as a requirement for the existence of a limit-cycle with period $T = \frac{2\pi}{\omega_0}$. On the other hand, the limit-cycle, if it exists, is symmetric, as shown below:

$$\begin{aligned} y(t+h) &= \sum_{n=-\infty}^{\infty} a_n e^{\jmath n\omega_0(t+h)} \\ &= \sum_{n=-\infty}^{\infty} a_n e^{\jmath n\omega_0 t}e^{\jmath n\omega_0 h} \\ &= \sum_{n=-\infty}^{\infty} a_n e^{\jmath n\omega_0 t}e^{\jmath n 2\pi h/T} \\ &= \sum_{n=-\infty}^{\infty} a_n e^{\jmath n\omega_0 t}e^{\jmath n\pi} \\ &= -y(t) \quad \forall t \end{aligned}$$

because $e^{\jmath n\pi} = -1$ for odd values of n, and all terms corresponding to even values of n are zero; this also implies that if $y(0) = 0$ then $y(h) = 0$. Hence, condition (29) implying that the output of the plant changes signal at $t = 0$, thus matching the first switching assumed for the square wave (23), also implies that all assumed switching times of the square wave are also matched by signal changes of the plant's output. But for the square wave to exist, there must be no additional signal changes of the plant's output, so one must have

$$y(t) = \sum_{n=-\infty}^{\infty} \jmath\frac{4}{\pi n}G(\jmath n\omega_0)e^{\jmath n\omega_0 t} < 0 \quad \forall t \in (0, \pi/\omega_0) \tag{30}$$

With this we have proven the following result, which is instrumental in determining the existence of SULCs.

Theorem 3 *The relay feedback system defined in (1)(11) presents a SULC with period $T = \frac{2\pi}{\omega_0}$ if and only if ω_0 satisfies equations (29) and (30).*

◇

Clearly a necessary condition for the satisfaction of (29) is that the imaginary part of the frequency response must change sign as ω goes from 0 to ∞; if all terms of the sum have the same sign they can not add up to zero. This leads to the following corollary of Theorem 3.

Corollary 1 *If the imaginary part of the frequency response does not change sign, that is:*

$$Im(G(\jmath\omega_1))Im(G(\jmath\omega_2)) \geq 0 \quad \forall\omega_1, \omega_2 > 0 \tag{31}$$

then there is no SULC in the relay feedback system (1)(11).

◇

This condition excludes, among other classes of plants, those that are stable and minimum-phase with order smaller than three.

4 Motivational example

We have just presented continuous-time tools for the analysis of relay feedback systems. Let us illustrate the typical performance of these analysis tools in the presence of sampling by means of an example. Consider a plant whose transfer function is

$$G(s) = \frac{1}{(s+1)(2s+1)(10s+1)}. \tag{32}$$

The analysis tools seen in Section 3 have been applied to this system. Application of Theorem 2 requires picking a minimal realization of the transfer function (32) and then finding the solutions of equation (21). We have chosen the canonical controllable realization and we have solved (21) numerically with MatLab, finding the unique solution $h = 3.975\ s$, which was then verified to satisfy equation (22). Therefore, one and only one SULC with period $T = 7.950\ s$ is predicted. The corresponding amplitude of the output can also be predicted, by simulating the plant with the initial condition $\mathbf{x}(0) = \mathbf{x}^*$, which yields $y_{max} = 0.066$.

In order to apply the frequency response analysis in Theorem 3, the series (29) must be approximated by a finite sum, and the result will naturally depend on the number of terms kept in this approximation. Taking the first ten terms (that is, $n = 1, 3, \ldots, 19$) leads to the same four significant digits as the exact time domain analysis, and only three terms ($n = 1, 3, 5$) are required to obtain three correct significant digits. Recall that describing function analysis is, in a sense, a particular case of our analysis, which is equivalent to taking only the first harmonic approximation in (29). In this example, which is particularly well-behaved, describing function analysis predicts a SULC with half-period $h = 3.897\ s$, a difference of only 2% with respect to the exact value.

Now suppose this relay feedback experiment is implemented digitally, as in Figure 2. If the sampling rate is high enough then the prediction done by continuous time analysis should provide sufficiently accurate results. Indeed, simulating the sampled-data system in SIMULINK with a sampling period $T_s = 0.001\ s$ a SULC is observed with the predicted values of period and amplitude with three coincident significant digits. For larger sampling periods one can expect that the behaviour of the sampled-data will be considerably different, and indeed it differs considerably even at sensible sampling rates. For $T_s = 0.1\ s$, for instance, the period and amplitude of the limit-cycle observed in simulation are somewhat different from the ones predicted, and depend on the initial condition: a period $h = 4.2\ s$ with amplitude $y_{max} = 0.074$ is observed for some initial conditions, whereas for other initial conditions values much closer to the ones predicted are observed.

For $T_s = 1\ s$ quite significant differences arise, with an amplitude $y_{max} = 0.148$ being observed for some initial conditions. As a matter of fact three different SULCs can be observed in simulation, depending on the initial conditions. The plant's output for each one of these SULCs is plotted in Figure 3, along with the output of the plant for the continuous time (that is, not sampled) SULC.

Tuning rules for PID controllers are usually such that the proportional gain is inversely proportional to the amplitude of the oscillation, whereas the integral time and the derivative time are proportional to the period [1]. So, tuning a PID based on the results of an experiment such as the one just presented could result in any one of three very different settings, depending on which SULC is observed, which in its turn is not something under the designer's control. Only one of these settings (the one resulting from the amplitude and period closer to the ones predicted by the continuous time) would be close to the one recommended by the theory, and there is no reason to believe that the other settings would be appropriate.

Note that these sampling periods are well within the range usually recommended for this plant in control systems. Indeed, even $T_s = 1\ s$ gives forty samples within the plant's rising time, compared to ten samples per rising time usually recommended in textbooks [2]. Also, the corresponding Nyquist frequency is given by $\omega_N \stackrel{\Delta}{=} \pi/T_s = \pi\ rad/s$, which is much above the plant's bandwidth, since $\mid G(\jmath\omega_N) \mid = 0.0015$. So, even for reasonable sampling rates, the effect of the sampler must be taken into account in the analysis. In this paper we provide a discrete-time analysis of limit-cycles that will provide more precise results.

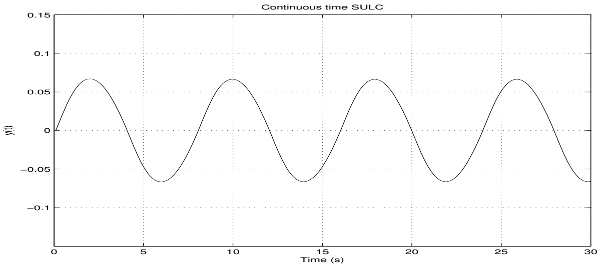
5 Discrete time analysis

In this Section the behaviour of the sampled feedback connection (4)(11) (equivalently, (5)(6)(11)) is analyzed in the discrete-time domain, reviewing and expanding the analysis presented in [3].

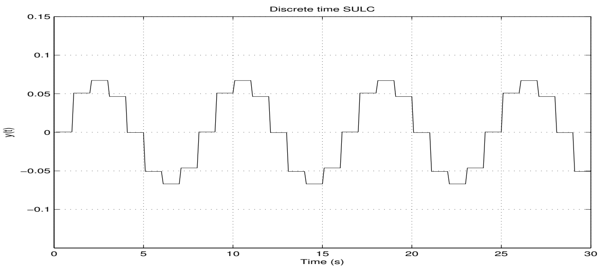
5.1 Exact time domain analysis

Assume that there is a symmetric unimodal oscillation with period $N = 2M$, with integer M, and recall the definitions of switching surface and of the half-spaces in (13), (14) and (15). Because of sampling, the relay will not switch when the state crosses the switching surface $\mathcal{S}$, but only at the first sample after $\mathcal{S}$ has been crossed. Accordingly, let $\mathbf{x}^*$ be the state at which this switching occurs, which corresponds to the first sample reached by the limit-cycle after crossing $\mathcal{S}$ from $\mathcal{X}_-$ to $\mathcal{X}_+$. Then, starting from the initial condition $\mathbf{x}_0 = \mathbf{x}^*$, the system behaves according to:

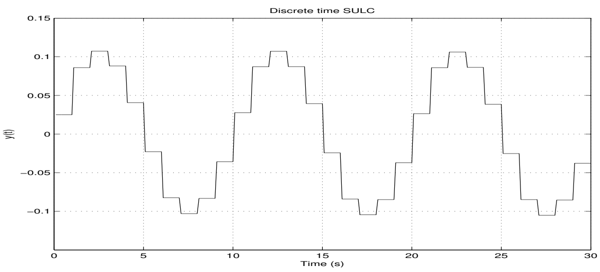
$$\mathbf{x}_{i+1} = \Phi\mathbf{x}_i - \Psi \quad \mathbf{x}_0 = \mathbf{x}^*. \tag{33}$$



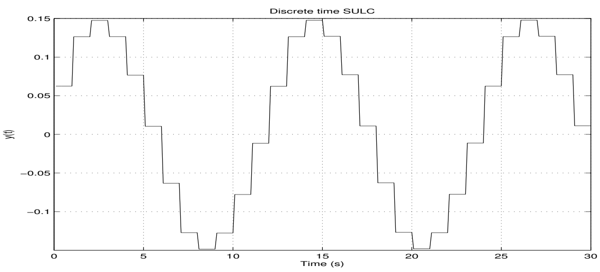
(a) Continuous time - $T = 7.95\ s$



(b) Sampled - $T = 8\ s$



(c) Sampled - $T = 10\ s$



(d) Sampled - $T = 12\ s$

Fig. 3 Time response of the relay feedback system: continuous time in (a); sampled with $T_s = 1\ s$ for different initial conditions in (b), (c) and (d)

The solution of the dynamic equation (33) is [3]

$$\mathbf{x}_i = \Phi^i \mathbf{x}^* - \sum_{j=0}^{i-1} \Phi^j \Psi$$
$$= \Phi^i \mathbf{x}^* - (\Phi - I)^{-1}(\Phi^i - I)\Psi \qquad (34)$$

[3] Once more, the matrix inversion is there only for convenience of notation

Because the oscillation is symmetric, $\mathbf{x}_M = -\mathbf{x}^*$ and thus

$$\mathbf{x}_M = -\mathbf{x}^* = \Phi^M \mathbf{x}^* - (\Phi - I)^{-1}(\Phi^M - I)\Psi \qquad (35)$$

Isolating $\mathbf{x}^*$ in (35) yields:

$$\mathbf{x}^* = (\Phi^M + I)^{-1}(\Phi^M - I)(\Phi - I)^{-1}\Psi \qquad (36)$$

So, if there is a SULC with period $2M$ then it must pass through $\mathbf{x}^*$ given in (36) and through the states given by (34) for $i = 1, 2, \ldots, M-1$; moreover, all these points $\mathbf{x}_i$ must be on the same side of the switching surface, that is, $C\mathbf{x}_i > 0,\ i = 0, 1, \ldots, M-1$. On the other hand if these conditions are satisfied, then the corresponding trajectory is a SULC by definition. With this we have proven the following result.

Theorem 4 *Let $\mathbf{x}^*$ be given by (36) and $\mathbf{x}_i$ by (34); also let M be and integer and $\mathbf{x}_0 = \mathbf{x}^*$. There exists a SULC with period $2M$ in the system (4) - (11) if and only if*

$$C\mathbf{x}_i > 0 \quad i = 0, 1, \ldots, M-1. \qquad (37)$$

◇

From Theorem 4 one can conceive an algorithm to search for all limit-cycles in a given relay feedback system.

Algorithm
For $M = 1, \ldots, \bar{M}$

1. set $L = 1$
2. calculate $\mathbf{x}^*$ from (36)
3. For $i = 1, \ldots, M-1$
   (a) calculate $\mathbf{x}_i$ from (34)
   (b) if $C\mathbf{x}_i < 0$ then $L = 0$
4. if $L = 1$ then there is a SULC with period $N = 2M$ else there is no SULC with period $N = 2M$

5.2 Relating to the continuous-time

Let us compare the conditions just obtained with their continuous-time counterparts, presented in subsection 3.2. The switching state in discrete-time is given in (36) which, after substitution of (7)(9) yields:

$$\mathbf{x}^* = (\Phi^M + I)^{-1}(\Phi^M - I)(\Phi - I)^{-1}\Psi$$
$$= (e^{AMT_s} + I)^{-1}(e^{AMT_s} - I)A^{-1}B \qquad (38)$$

This is the same expression as (20), which describes the switching state in the continuous-time case, with h replaced by MT_s. So, the switching state in a given

SULC is given by the same expression whether the system is sampled or not. But this does not imply an equivalence between the two situations, as the existence of a SULC requires the switching state to satisfy different conditions in each setting. In the continuous-time case the switching state must belong to the switching surface, whereas in the discrete-time case it only has to be "close" to it. More specifically, some convex combination $\bar{\mathbf{x}}$ of the switching state and the previous system's state must belong to the switching surface, that is:

$$\begin{aligned} C\bar{\mathbf{x}} &= C[\lambda \mathbf{x}^*(h) + (1-\lambda)\Phi^{-1}\mathbf{x}^*(h)] \\ &= C[\lambda I + (1-\lambda)\Phi^{-1}]\mathbf{x}^*(h) \\ &= C[\lambda(I-\Phi^{-1}) + \Phi^{-1}]\mathbf{x}^*(h) = 0 \end{aligned} \quad (39)$$

for some $\lambda \in [0,1]$. So, existence of a SULC with period h in the continuous time case requires that h is a positive real number satisfying (39) for $\lambda = 1$. In the discrete-time case, existence of a SULC with period h requires that h satisfies (39) for any $\lambda \in (0,1)$, but this h must be an integer multiple of T_s. As a consequence of the fact that the switching state is less constrained, a SULC present in continuous time typically turns, when the system is sampled, into several SULCs with "neighbour" periods, as seen in the case study in Section 4.

5.3 Exact frequency domain analysis

If a symmetric unimodal limit-cycle with period $N = 2M$ is observed in the discrete-time relay feedback system (4)(11), then the linear system's input is a square wave with period N. If we take an instant at which the square wave switches up - that is, from -1 to $+1$ - as the time reference, then this square wave is described by:

$$\begin{aligned} u(k) &= +1 \qquad k = 1+mN, 2+mN, \ldots, M+mN \\ u(k) &= -1 \qquad k = M+1+mN, \ldots, N+mN \end{aligned} \quad (40)$$

for all $m \in \mathbb{Z}$. The expansion of the square wave (40) in Fourier series yields:

$$u(k) = \frac{1}{N}\sum_{n=1}^{N} c_n e^{\jmath n\Omega_0 k} \quad (41)$$

where $\Omega_0 = \frac{2\pi}{N}$ and the coefficients c_n are given by

$$c_n = \frac{2\jmath\, e^{-\jmath\pi n/N}}{\sin(\frac{\pi}{N}n)} \quad (42)$$

for odd values of n and $c_n = 0$ otherwise - the DC component is zero, as well as all even harmonics. [4]

[4] This expression is deduced in the Appendix.

The input of the plant is the output of the relay, which switches up when its input changes signal from positive to negative. But the relay's input is the plant's output, so $u(k)$ will be given by (40) if and only if the plant's output changes signal at the same time instants and in the opposite direction, that is,

$$\begin{aligned} & y(0) > 0 \\ & y(k) < 0 \ k = 1, \ldots, M \\ & y(k) > 0 \ k = M+1, \ldots, N \\ & y(k) < 0 \ k = N+1, \ldots, N+M \end{aligned}$$

and so on. If the input of the plant is the square wave (41), then its output is given by

$$\begin{aligned} y(k) &= \frac{1}{N}\sum_{n=1}^{N} G(e^{\jmath n\Omega_0}) c_n e^{\jmath n\Omega_0 k} \\ &= \frac{2\jmath}{N}\sum_{n=1\ odd}^{N} G(e^{\jmath n\Omega_0})\frac{e^{-\jmath\pi n/N}}{\sin(\frac{\pi}{N}n)} e^{\jmath n2\pi k/N} \\ &= \frac{\jmath}{M}\sum_{n=1\ odd}^{2M} G(e^{\jmath n\pi/M})\frac{e^{-\jmath\pi n/2M}}{\sin(\frac{\pi}{2M}n)} e^{\jmath n\pi k/M} \end{aligned} \quad (43)$$

Let us note that the limit-cycle, if it exists, is symmetric, that is

$$\begin{aligned} y(k+M) &= \frac{2\jmath}{N}\sum_{n=1\ odd}^{N} G(e^{\jmath n\Omega_0})\frac{e^{-\jmath\pi n/N}}{\sin(\frac{\pi}{N}n)} e^{\jmath n2\pi(k+M)/N} \\ &= \frac{2\jmath}{N}\sum_{n=1\ odd}^{N} G(e^{\jmath n\Omega_0})\frac{e^{-\jmath\pi n/N}}{\sin(\frac{\pi}{N}n)} e^{\jmath n2\pi k/N} e^{\jmath n\pi} \\ &= -\frac{2\jmath}{N}\sum_{n=1\ odd}^{N} G(e^{\jmath n\Omega_0})\frac{e^{-\jmath\pi n/N}}{\sin(\frac{\pi}{N}n)} e^{\jmath n2\pi k/N} \\ &= -y(k) \end{aligned}$$

since $e^{\jmath n\pi} = -1$ for odd values of n.

On the other hand, from the symmetry of the coefficients c_n and of the transfer function $G(e^{\jmath n\Omega_0})$, we can write

$$\begin{aligned} y(k) &= \sum_{n=1\ odd}^{M} 2Re\{\frac{\jmath}{M} G(e^{\jmath n\pi/M})\frac{e^{-\jmath\pi n/2M}}{\sin(\frac{\pi}{2M}n)} e^{\jmath n\pi k/M}\} \\ &= \frac{-2}{M}\sum_{n=1\ odd}^{M} \frac{1}{\sin(\frac{\pi}{2M}n)} \times \\ &\qquad Im\{G(e^{\jmath n\pi/M}) e^{-\jmath\pi n(k-0.5)/M}\} \end{aligned} \quad (44)$$

With this we have proven the following result.

Theorem 5 *Let $y(k)$ be given by (44). There exists a SULC with period $N = 2M$ if and only if M satisfies* *the following set of $M+1$ inequalities:*

$$y(0) > 0$$
$$y(k) < 0 \; k = 1, \ldots, M \tag{45}$$

◇

The existence of SULCs in a given relay feedback system can be verified by checking conditions (45) for all values of M in a "reasonable" range, that is, up to a large enough value.

5.4 First-order approximation

Insight is gained by looking at a first-harmonic approximation of the output signal $y(k)$, which is given by

$$y(k) \approx \frac{-2}{M} \frac{1}{\sin(\frac{\pi}{2M})} Im\{G(e^{\jmath\pi/M})e^{-\jmath\pi(k-0.5)/M}\} \tag{46}$$

which for $k=0$ gives

$$y(0) \approx \frac{-2}{M} \frac{1}{\sin(\frac{\pi}{2M})} Im\{G(e^{\jmath\pi/M})e^{\jmath\pi/2M}\}$$

and for $k=1$ gives

$$y(1) \approx \frac{-2}{M} \frac{1}{\sin(\frac{\pi}{2M})} Im\{G(e^{\jmath\pi/M})e^{-\jmath\pi/2M}\}.$$

To satisfy the constraints that $y(0) > 0$ and $y(1) < 0$, one must have

$$Im\{G(e^{\jmath\pi/M})e^{\jmath\pi/2M}\} < 0$$

and

$$Im\{G(e^{\jmath\pi/M})e^{-\jmath\pi/2M}\} > 0$$

which implies that

$$arg(G(e^{\jmath\pi/M})) \in (-\pi - \pi/2M, -\pi + \pi/2M), \tag{47}$$

that is, the oscillation is at a frequency for which the phase of the transfer function lies around $-\pi$, in an interval of size $-\pi/N$. Thus, it is reasonable to use as an approximation for the prediction of SULCs any period M - equivalently $\Omega = \pi/M$ - for which the phase of the plant's frequency response $arg(G(e^{\jmath\Omega}))$ is close to $-\pi$, in a similar way to the describing function approach for continuous-time systems.

It is worth noting that the phase of a strictly proper discrete-time transfer function always reaches $-\pi$ for some finite frequency. So, we conjecture that all sampled transfer functions will satisfy condition (47) for some M and thus exhibit at least one SULC.[5] On the other hand, the condition in Corollary 1 excludes a huge class of plants, showing that for these plants the continuous-time analysis erroneously predicts that there will be no SULCs. For example, it is quite easy to see that a strictly proper first-order BIBO-stable $G(s)$ does not satisfy condition (31), but the corresponding $G(z)$ satisfies (47) with $M=1$ for any sampling period and thus the sampled relay feedback system with such a $G(s)$ does exhibit a SULC.

[5] Though we have not yet been able to prove it.

6 The motivational example revisited

Let us analyse the behaviour of the Example in Section 4 with the tools just developed. Recall that for $T_s = 1$ s three different SULCs have been observed in simulation, as shown in Figure 3, and that the continuous time analysis was unable to predict this behaviour correctly. For $T_s = 1$ s the sampled transfer function is given by:

$$G(z) = 10^{-3} \frac{5.671z^2 + 15.45z + 2.55}{z^3 - 1.879z^2 + 1.105z - 0.2019} \tag{48}$$

and the corresponding sampled state space representation is:

$$\Phi = \begin{bmatrix} 0.0845 & -0.3058 & -0.0224 \\ 0.4475 & 0.8004 & -0.0149 \\ 0.2983 & 0.9247 & 0.9943 \end{bmatrix}$$
$$\Psi = \begin{bmatrix} 0.4475 & 0.2983 & 0.1134 \end{bmatrix}^T. \tag{49}$$

Start with the first harmonic approximation given by condition (47). The frequency response of the transfer function (48) is presented in Figure 4, where the frequencies corresponding to integer even periods are highlighted. The phase intervals described in equation (47) are also depicted in this plot, showing that (47) is satisfied for several values of N, namely $N=8$, $N=10$ and $N=12$; these are exactly the periods observed in simulation. The amplitudes of each SULC can be estimated by the amplitude in equation (46), which yields $A_8 = 0.0679$, $A_{10} = 0.1054$ and $A_{12} = 0.1459$ (A_N standing for the amplitude of the SULC with period N). Thus, from the first harmonic approximate analysis one can expect the occurrence of three SULCs with the periods and amplitudes listed above, which is quite close to what is observed in simulation. It turns out that for this particular example the first harmonic approximation provides a good estimate, predicting all the SULCs with the correct periods and with at least two significant digits for the amplitudes. Given that the first harmonic condition can be easily checked visually with the aid of the plant's frequency response, it is a very convenient tool to be preferred whenever possible.

Now, let us apply the exact conditions of Theorem 5 to this case, with no approximations. To do this, condition (45) must be checked for $G(z)$ in (48) with all

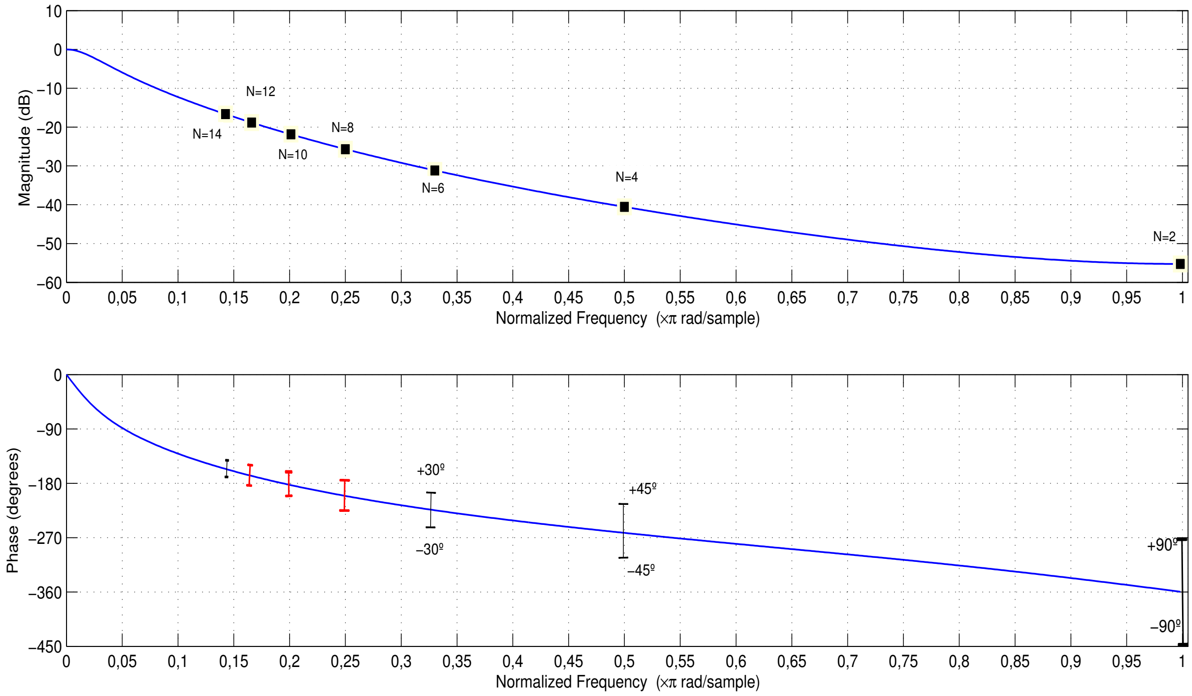


Fig. 4 Frequency response of (48); dots in the magnitude graph represent frequencies corresponding to integer even periods; the ranges defined in equation (47) are shown in the phase graph; those frequencies for which the range includes -180^o (in red) correspond to the existing SULCs

positive integers M. We have checked these conditions for all $M \leq 100$, and they are satisfied for three different values of M in this range: $M = 4$, $M = 5$ and $M = 6$, confirming the findings of the first harmonic approximation. The predicted amplitudes are slightly different and closer to the real values (those observed in simulation): $A_8 = 0.0671$, $A_{10} = 0.1055$ and $A_{12} = 0.1480$.

Finally, consider the application of Theorem 4 to this example. Verifying (37) results once again in a positive answer for $M = 4$, $M = 5$ and $M = 6$ - that is $N = 8$, $N = 10$ and $N = 12$. The corresponding values of the switching state are given by

$$\mathbf{x}_8^* = \left[0.2931\ 1.1174\ 0.0090\right]^T$$
$$\mathbf{x}_{10}^* = \left[0.1398\ 1.3686\ 0.5129\right]^T$$
$$\mathbf{x}_{12}^* = \left[0.0293\ 1.5053\ 1.2518\right]^T$$

and the amplitudes of the SULCs are the same as predicted by the exact frequency domain analysis.

Notice that the state space condition (37) is much easier to verify than the frequency domain condition (45). The complexity of checking (45) grows as M^2, since M inequalities must be checked and each one consists of a sum with M terms. Whereas (37) consists of M inequalities as well, but each one of the inequalities has the same complexity for all M. For the current example, which is of low order, checking all values of M up to one thousand already takes several minutes in a typical personal computer with the frequency response approach, but only a few seconds with the state space approach.

7 Conclusions

Continuous-time analysis of a relay feedback system is not appropriate to predict the existence of SULCs and their properties when the system is sampled. Relevant differences in the period and amplitude appear even for sampling rates that are quite fast with respect to the plant's time constant. A common phenomenon, which we have illustrated by a very ordinary example, is that several SULCs with significantly different periods and amplitudes appear in the sampled system where the continuous time analysis predicts only one.

We have given two exact methods to predict SULCs in sampled relay feedback systems. One method is based on the state space representation of the discrete time system and is very similar to the state-space methods known for continuous time systems. Confronting the two state-space approaches also explains the appearance of extra SULCs in the sampled system. We have also revisited frequency domain results from the 1960's, presenting them under a more contemporary light, which allows one to explore their application with today's computer methodologies, unavailable at the time this theory has been developed. The application of these methods to the example shows that they are easy to use and predict exactly all the symmetric unimodal limit-cycles in sampled relay feedback systems.

Among the many consequences of errors in prediction of SULCs by the continuous time analysis, one is of particular interest and appeal to the authors: Ziegler-Nichols-like tuning based on the results of sampled relay feedback experiments could fail. Hence, a matter of current research is the adaptation of Ziegler-Nichols-like tuning procedures and formulas to cope with this situation, based on the analytical methods for predicting SULCs in discrete-time systems presented in this paper.

A Deduction of the Fourier series of the square wave

The Fourier series is given by

$$\begin{aligned} c_n &= \sum_{k=1}^{N} u(k) e^{-\jmath n \Omega_0 k} \\ &= \sum_{k=1}^{M} e^{-\jmath n\pi k/M} - \sum_{k=M+1}^{N} e^{-\jmath n\pi k/M} \triangleq W - X \end{aligned}$$

Using the formula for the sum of the terms of a PG $\sum = a_1 \frac{1-q^m}{1-q}$ where q is the ratio, m is the number of terms and a_1 is the first term, the first summation is equal to:

$$W = \sum_{k=1}^{M} e^{-\jmath n\pi k/M} = e^{-\jmath n\pi/M} \frac{1-(e^{-\jmath n\pi/M})^M}{1-e^{-\jmath n\pi/M}}$$

Multiply the numerator and the denominator by $e^{+\jmath n\pi/2M}$ to get rid of the complex number in the denominator:

$$\begin{aligned} W &= e^{-\jmath n\pi/M} e^{\jmath n\pi/2M} \frac{1-(e^{-\jmath n\pi})}{e^{\jmath n\pi/2M} - e^{-\jmath n\pi/2M}} \\ &= e^{-\jmath n\pi/2M} \frac{1-(e^{-\jmath n\pi})}{e^{\jmath n\pi/2M} - e^{-\jmath n\pi/2M}} \\ &= e^{-\jmath n\pi/2M} \frac{1-(e^{-\jmath n\pi})}{-2\jmath \sin(n\pi/2M)} \end{aligned} \quad (50)$$

From now on we can proceed like in [14] and other textbooks in Fourier analysis, where it is customary to multiply the numerator by $e^{-\jmath n\pi/2} \times e^{+\jmath n\pi/2}$ to get

$$\begin{aligned} W &= e^{-\jmath n\pi/2M} e^{-\jmath n\pi/2} \frac{e^{+\jmath n\pi/2} - (e^{-\jmath n\pi/2})}{-2\jmath \sin(n\pi/2M)} \\ &= e^{-\jmath n(\pi/2M+\pi/2)} \frac{\sin(n\pi/2)}{\sin(n\pi/2M)} \end{aligned}$$

which is a common expression found in signal processing textbooks.

Or, alternatively, one can just recognise that the numerator in (50) is given by $1-(-1)^n$, which is either equal to 0 (for even values of n) or 2, for odd value of n, resulting in:

$$W = e^{-\jmath n\pi/2M} \frac{2}{-2\jmath \sin(n\pi/2M)} = \frac{\jmath e^{-\jmath n\pi/2M}}{\sin(n\pi/2M)} \quad (51)$$

for n odd and $W = 0$ for n even. For $n = 0$ there is an indetermination in W, but this will not be a problem in our calculations, as will be seen in the sequel. The second term of the coefficient c_n is

$$X = \sum_{k=M+1}^{N} e^{-\jmath n\pi k/M}$$

which is also a PG with the same ratio and the same number of terms as W, the only difference being the first term, which now is given by $e^{-\jmath n\pi(M+1)/M}$ instead of $e^{-\jmath n\pi/M}$. Thus it seems obvious that

$$X = \frac{e^{-\jmath n\pi(M+1)/M}}{e^{-\jmath n\pi/M}} W = e^{-\jmath n\pi} W = (-1)^n W$$

which gives $X = W$ for $n = 0$ or n even, and $X = -W$ for n odd. Since $c_n = W - X$, this gives $c_n = 0$ for $n = 0$ or n even, and $c_n = 2W$ for n odd. Finally, using the expression (51):

$$c_n = \frac{2\jmath e^{-\jmath n\pi/2M}}{\sin(n\pi/2M)}$$

for n even and $c_n = 0$ otherwise.

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