A new method for PID tuning including plants without ultimate frequency

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Abstract—The classical relay feedback method for tuning PID controllers cannot be applied to plants whose Nyquist diagrams do not cross the negative real axis; these are customarily tuned based on the reaction curve experiment. In this paper we propose a tuning method based on a modified relay feedback experiment. In this experiment, a transfer function of constant phase in an arbitrarily large range of frequencies is inserted in the loop. The proposed methodology thus unifies the Ziegler-Nichols-like tuning methods, by allowing PID tuning based on relay feedback for a class of plants without ultimate frequency.

I. INTRODUCTION

The seminal work \cite{12}, by J.G. Ziegler and N.B. Nichols, gave rise to two families of tuning methods and formulas for PID controllers: those based on the open-loop step response of the plant (also called its reaction curve) and those based on a closed-loop experiment. The second family applies only to plants with a finite ultimate frequency - otherwise stated, whose Nyquist diagram crosses the negative real axis. Hence, plants that do not have an ultimate frequency must be tuned by some method in the first family, but this is not always possible either - not all such plants have a reaction curve that allows the application of these methods.

In this paper we present a tuning method based on a modified relay feedback experiment, which can be applied also to plants that do not possess an ultimate frequency. It thus serves as an alternative method for tuning controllers for this class of plants, but it also enlarges the class of plants for which Ziegler-Nichols-like method can be applied. Moreover, it can also be applied to plants that do possess an ultimate frequency, and thus removes the need for a priori knowledge of the class the plant belongs to. Similar solutions have been presented previously, but they require more than one experiment, thus impairing considerably the method’s main assets, which are the simplicity, speediness and autonomy.

The paper is organized as follows. After some preliminaries, we describe in Section III the control design philosophy, which relies on the knowledge of a particular point of the plant’s frequency response. In the following section we present the modified relay feedback experiment that allows to obtain this information with just one experiment, which consists of the inclusion of a Fractional Order Integrator (FOI) in the loop. The efficacy of the proposed tuning methodology is shown by applying it to a wide variety of plants. The application to two real-life problems that are not amenable to either of the usual Ziegler-Nichols (ZN) methods is described in detail in Section V, and the results obtained in a large range of different plants are summarized in Section VI.

II. PRELIMINARIES

A. The plants

We consider linear time invariant causal (LTIC) plants which are described by

\[ Y(s) = G(s)U(s) \]  

where \( G(s) \) is the plant’s transfer function, \( U(s) \) and \( Y(s) \) are respectively the Laplace transforms of the control input and of the plant’s output (the controlled variable). The plant is controlled by an LTIC controller, that is:

\[ E(s) = R(s) - Y(s) \]  
\[ U(s) = C(s)E(s) \]

where \( R(s) \) is the reference, \( E(s) \) is the tracking error and \( C(s) \) is the controller’s transfer function.

B. The PI(D) controllers

The transfer function of a Proportional-Integral (PI) controller can be written as

\[ C_{PI}(s) = K_p(1 + \frac{1}{T_i s}) \]  

where \( T_i \) is called the integral time and \( K_p \) is the proportional gain; these are the parameters to be tuned. For the design of Proportional-Integral-Derivative (PID) controllers we consider the so-called series form, with ideal derivative action:

\[ C_{PID}(s) = K_p(1 + \frac{1}{T_i s})(1 + T_d s) \]  

Derivatives can not be implemented exactly in real life, so the transfer function of a real PID controller usually has the following form:

\[ C_{PID}(s) = K_p(1 + \frac{1}{T_i s})(1 + T_d s + \frac{s}{Ns + 1}) \]  

where \( N \) is a fixed parameter. In this paper the ideal transfer function (5) is used for design purposes, whereas in the simulations the real transfer function (6) is used. The parameter \( N \) must be small enough to guarantee that the approximation (5) to the real transfer function (6) is meaningful in the frequency range of interest. The smaller the \( N \), the better this approximation, but there are lower limits for \( N \) dictated by considerations outside the scope of our paper, most notably the level of noise (because the high frequency gain of the controller is \( (T_d/N + 1)K_p \)) and the parametric precision of the implementation. In all simulations to be presented we have set \( N = \frac{10}{\omega_{120}} \), where \( \omega_{120} \) is defined in Section III.

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C. The classical forced oscillation method

Huge amounts of literature have been produced on PID tuning rules, and a myriad of methods have been proposed and successfully applied - see [2] for a thorough overview. Extensions of these methods continue to appear, including multivariable plants [4], resonant controllers [8] and event-based controllers [3], to name a few. These methods and extensions consist in variations of the methods proposed in the seminal work [12]. In [12], a tuning method was proposed that consists in causing an oscillation in closed-loop, measuring the oscillations’ frequency and amplitude and then applying simple formulas for each controller parameter, formulas that involve these measurements. In another seminal work [1], this method has been improved and reinvigorated by the relay feedback experiment and the explicit consideration of gain and phase margins. In this paper we shall refer to this method as the classical forced oscillation (CFO) method.

The CFO method is based solely on the knowledge of the ultimate point of the plant’s frequency response. The ultimate point for a given transfer function is the point at which its Nyquist plot crosses the negative real axis. The characteristics of the ultimate point are the ultimate frequency \( \omega_u \) and the ultimate gain \( K_u \), which are defined as:

\[
\omega_u = \min_{\omega \geq 0} \omega : \angle G(j\omega) = -\pi
\]

\[
K_u = \frac{1}{|G(j\omega_u)|}
\]

With these definitions, the CFO method can be summarized as follows.

1) identify the ultimate point of the plant’ frequency response, that is, determine \( \omega_u \) and \( K_u \);
2) choose the parameters of the controller such that

\[
C(j\omega_u) = -K_u p,
\]

where \( p \) is a prespecified location in the complex plane.

The first step of the method is usually performed by means of a relay feedback experiment, which consists in a closed-loop experiment with the following nonlinear control action:

\[
u(t) = d \text{sign}(e(t)) + b.
\]

In (8) \( \text{sign}(\cdot) \) is the sign function (\( \text{sign}(x) = 1 \) for positive \( x \) and \( \text{sign}(x) = -1 \) for negative \( x \)), \( d \in \mathbb{R}^+ \) is a parameter to be chosen and \( b \in \mathbb{R} \) is the bias. The bias parameter \( b \) must be adjusted so that the oscillation is symmetric. Once a symmetric oscillation is obtained, its amplitude \( A_u \) and period \( T_u \) are measured and the ultimate quantities are calculated from [2]

\[
K_u = \frac{4d}{\pi A_u} \quad \omega_u = \frac{2\pi}{T_u}
\]

The second step of the method is accomplished by solving equation (7) for the controller’s gains \( K_p, T_p, T_d \) with the chosen location \( p \). Under the reasonable assumption that the frequency response of the plant is sufficiently smooth, shifting the ultimate point away from \(-1\) in the complex plane implies shifting the whole frequency response away from it, thus leading to good stability margins. Different locations \( p \) have been proposed over the years, each one providing different transient performance and stability margins. The original Ziegler-Nichols tuning formulas in [12] correspond to \( p = -0.4 + j0.08 \) for PI controllers and \( p = -0.6 - j0.28 \) for PID controllers.

Plants that do not possess an ultimate point are not amenable to application of this method. This is the case of all minimum-phase stable second-order plants and most plants with relative degree smaller than three, for instance. A method based on relay feedback for larger classes of plants has been proposed in [6], where more than one experiment is required, and at each experiment a designer must intervene to adjust the parameters of the next experiment. Other similar methods have also been proposed, with the same virtues (being applicable to larger classes of plants) and limitations (requiring more complex experiments).

Our method, to be presented next, is based on the same theoretical justification as the CFO method: placing one particularly relevant point of the loop frequency response at a specified location in the complex plane, which guarantees appropriate stability margins provided that the plant’s frequency response is sufficiently smooth. It can be applied to all plants with relative degree larger than one, thus being applicable to a much broader class of plants than the CFO. And, unlike [6] and other similar solutions, it does not require extra experiments and/or intervention of the designer. We have baptized it the Extended Forced Oscillation (EFO) method.

III. THE EXTENDED FORCED OSCILLATION (EFO) METHOD

The control design objective of the CFO method is to obtain an appropriate stability margin, that is, gain margin and/or phase margin [1]. If the plant’s frequency response does not cross the negative real axis, then the gain margin will be infinite, provided that the controller does not contribute with too large a phase delay. So, in extending the CFO method for this class of plants, the phase margin becomes the only explicit control design objective. Following the CFO method’s theoretical approach, we propose to identify the point of the plant’s frequency response for which the desired phase margin is achieved and then design a controller such that: i) the controller’s contribution to the phase at this particular frequency is small; ii) the magnitude of the loop transfer function at this frequency is unitary.

Let us now develop this idea analytically. Let \( M_\theta \) be the desired phase margin and \( \theta = 180^\circ - M_\theta \). Identify the frequency \( \omega_0 \) defined as \( \angle G(j\omega_0) = \theta \), and the magnitude of the plant’s frequency response at this frequency: \( M_\theta = |G(j\omega_0)| \). We will present in the next Section a relay feedback experiment that yields this information; for the moment assume that this information - that is, \( \omega_0 \) and \( M_\theta \) - has somehow become available. If the controller is designed such that

\[
C(j\omega_0)G(j\omega_0) = 1\angle \theta
\]

then the phase margin will be exactly the desired one, provided that the magnitude of the loop transfer function decreases monotonically for frequencies higher than \( \omega_0 \). Hence the
controller must be designed to satisfy (10) or, equivalently:

\[ C(j\omega) = \frac{1}{M_0} \angle 0^\circ \quad (11) \]

It remains to specify what would be a reasonable phase margin to choose. Control textbooks suggest values around 45° to provide appropriate robustness and dynamic performance for typical practical situations [11]. Here we must design the controller without a plant model, knowing only one point of the frequency response, so a larger specification for phase margin would be in order. We have designed, for a wide array of plants, PI and PID controllers with different specifications of phase margin and evaluated the resulting performance from the closed-loop reference step response. \(^1\)

From these tests the choice \( M_0 = 60^\circ \) (corresponding to \( \theta = -120^\circ \)) has emerged as the best one, so this is the value proposed. In the following we detail the use of (11) to obtain tuning formulas for PI and PID controllers.

A. PI

It is clear from (4) that \( \angle C_{PI}(j\omega) < 0 \forall \omega \), thus it is not possible to satisfy (11) exactly with a PI controller. In this case, we will have to content with designing the controller such that it satisfies (11) only approximately, that is, such that:

\[ C(j\omega) = \frac{1}{M_0} \angle -\beta. \quad (12) \]

with \( \beta \) a small positive angle. The classical formulas of the CFO method for PI controllers correspond to a \(-10^\circ\) contribution of the controller at the ultimate frequency. We adopt a similar criterion here, that is, we chose \( \beta = 10^\circ \).\(^2\)

With this choice, and the controller’s transfer function (4), equation (12) particularizes to

\[ C(j\omega_{120}) = K_p - j \frac{K_p}{\omega_{120} T_i} = \frac{1}{M_{120}} - 10^\circ \]

Equating the real and imaginary parts of this equation yields the tuning formulas proposed for a PI controller in this paper:

\[ K_p = \frac{\cos(10^\circ)}{M_{120}} \quad T_i = \frac{1}{\omega_{120} \tan(10^\circ)} = \frac{T_{120}}{2\pi \tan(10^\circ)} \quad (13) \]

B. PID

The inclusion of a PD block in the controller’s transfer function, as in (5), allows to achieve the objective of zero phase lag contribution by the controller at the identified frequency, as desired - that is, to satisfy equation (11) exactly. Indeed, the PD block \( 1 + T_d s \) can provide a phase lead to compensate for the phase lag inserted by the PI block \( 1 + \frac{1}{T_i} \). Let us start with a PI tuned by the formulas (13); then the PI block inserts a phase lag of \( 10^\circ \) at the identified frequency \( \omega_{120} \). To compensate for this lag, obtaining a controller that inserts no delay in the loop at the frequency \( \omega_{120} \), we must have

\[ \angle 1 + j\omega_{120} T_d = \arctan(\omega_{120} T_d) = 10^\circ, \]

\(^1\)The plants and the performance measures are those in Section VI

\(^2\)Different choices of \( \beta \) have been tried, and none has yielded better results than \( \beta = 10^\circ \).

TABLE I

<table>
<thead>
<tr>
<th>Controller</th>
<th>( K_p )</th>
<th>( T_i )</th>
<th>( T_d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>PI</td>
<td>0.99</td>
<td>0.90</td>
<td>0.12</td>
</tr>
<tr>
<td>PID</td>
<td>0.99</td>
<td>0.90</td>
<td>0.028</td>
</tr>
</tbody>
</table>

that is,

\[ T_d = \frac{\tan 10^\circ}{\omega_{120}} = \frac{\tan 10^\circ}{\pi T_{120}} \]

On the other hand, the PD block increases the magnitude of the controller’s transfer function by a factor

\[ \sqrt{1 + (T_d \omega_{120})^2} = \sqrt{1 + \tan^2 10^\circ} = \frac{1}{\cos 10^\circ}. \quad (14) \]

In order to keep the magnitude of the controller’s transfer function at the desired value unaltered, thus satisfying equation (11), the controller’s gain \( K_p \) must be reduced by this same factor. This leads to the new tuning for the proportional gain

\[ K_p = \frac{\cos^2(10^\circ)}{M_{120}} \]

which is the value given in (13) divided by the factor (14).

The formulas for the tuning of PI and PID controllers are summarized in Table I, where the trigonometric functions have been rounded up to two significant digits.

IV. FINDING OF THE \( \omega_{120} \)-POINT

The relay feedback experiment described in Section II-C is a classical tool to determine experimentally the ultimate point of a plant [2]. A natural extension of the relay feedback experiment allows the identification of other points of the plant’s frequency response. If a known transfer function, say \( F(s) \), is inserted in the loop in addition to the relay, as in Figure 1, then the oscillation will occur at the ultimate frequency of the transfer function \( F(s)G(s) \), that is, at \( \omega_1 : \angle F(j\omega_1)G(j\omega_1) = -180^\circ \). Then the plant’s frequency response at this frequency can be calculated as

\[ |G(j\omega_1)| = \frac{\pi A}{4d |F(j\omega_1)|} \quad \angle G(j\omega_1) = -180^\circ - \angle F(j\omega_1) \quad (15) \]

since \( F(j\omega_1) \) is known.

![Fig. 1. Relay feedback experiment for identification of the ultimate point of \( F(s)G(s) \).](image)

In order to implement the tuning method described in the previous Section, we need to identify the point of the plant’s frequency response at which the phase reaches a prespecified value: namely \( M_0 = 180^\circ \). For that, we need to insert in the loop a transfer function \( F(s) \) whose frequency response reaches \(-M_0 \) at this same frequency, but this frequency is not known in advance - on the contrary, it is one of the two
quantities that the experiment aims at identifying. An iterative set of experiments can be performed, each time with a different transfer function in the loop, until the “right” \( F(s) \) is found and the oscillation occurs at the desired frequency. Procedures like this have been proposed and applied before, like in [6], but in so doing the main assets of the CFO method - its simplicity and rapidity - are significantly impaired.

If the phase of the transfer function \( F(s) \) were the same for all frequencies, with \( \angle F(j\omega) = -M_\phi \forall \omega \), then only one experiment would be necessary, keeping the exact same coefficients given in Table II.

\[
F(s) = \frac{1}{s^m}, \quad (16)
\]

which can be easily seen as follows

\[
\angle F(j\omega) = -\angle(j\omega)^m = -\angle e^{m/2} = -\frac{\pi}{2} m. \quad (17)
\]

Putting \( m = \frac{M_\phi}{90^\circ} \) in (17) yields \( \angle F(j\omega) = -M_\phi \forall \omega \), as desired. In particular, for \( M_\phi = 60^\circ \) as proposed, \( m = \frac{2}{3} \).

### A. Implementation of the fractional order integrator

Practical implementation of fractional order systems is not as direct and well established as that of integer order systems. Most commonly, fractional order systems are implemented approximately by integer order systems. In the implementation described in Section VI we have used the MatLab package FOMCON [10], [9], to obtain a transfer function which approximates the magnitude and phase characteristics of the desired FOI. The approximation results in a transfer function with magnitude characteristics of \(-13.33 \text{ dB/dec}\) and a constant phase value of \(-60^\circ\) for a large range of frequencies (from \(10^{-3}\) to \(10^3\) rad/s), which is described by (18) with the coefficients given in Table II.

\[
F(s) = \sum_{k=0}^{11} b_k s^k + \sum_{n=0}^{11} a_n s^n \quad (18)
\]

Clearly, other options of FOIs could be used to accommodate different phase margin specifications, choosing the appropriate \( m \) and obtaining a finite order approximation for the FOI in the same way. Figure 2 presents the magnitude and phase curves of three such FOIs approximations, with \(-13.33 \text{ dB/dec}\) and \(-60^\circ\) (the one proposed in this paper), \(-10.00 \text{ dB/dec}\) \(-45^\circ\) and \(-6.66 \text{ dB/dec}\) and \(-30^\circ\).

### V. Two Case Studies

In order to validate the proposed PI/PID tuning method, first two different plants will be considered. For these two plants, a detailed analysis will be done, describing all the steps of the design. The first plant represents a linearized model of an aircraft pitch angle dynamics described in [7], which is a non auto regulated system with transfer function presented in (19).

\[
G(s) = \frac{1.15s + 0.18}{s^3 + 0.74as^2 + 0.92s} \quad (19)
\]

The frequency response of the plant described by equation (19) is presented in Figure 3. In order to tune the PI/PID controller’s gains following the conventional Ziegler-Nichols-like methods, first the open loop step response procedure is obtained, resulting in the output presented in Figure 4(a). Clearly, this is a non auto regulated plant, so tuning based on reaction curve methods can not be applied. Then a closed loop relay feedback experiment is performed, aiming at the application of the CFO method. The result of this experiment is shown in Figure 4(b), where it is seen that the CFO method can not be applied because the self oscillation condition was not satisfied.

As it is not possible to determine the PI/PID controller gains through either classical Ziegler-Nichols tuning procedures, the EFO method will be employed. Figure 5(a) shows the self oscillatory behavior of the system working in closed loop with a relay and a FOI, as in Figure 1. Through this experiment

![Bode Diagram](image-url)

**Fig. 2.** Frequency response of the FOI approximations with \(-60^\circ\) in red, \(-45^\circ\) in blue and \(-30^\circ\) in green.

### TABLE II

<table>
<thead>
<tr>
<th>( i )</th>
<th>( a_i )</th>
<th>( b_i )</th>
</tr>
</thead>
<tbody>
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<td>0</td>
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<td>0.7152</td>
</tr>
<tr>
<td>1</td>
<td>11.13</td>
<td>1.445 x 10^4</td>
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<tr>
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<td>4.387 x 10^5</td>
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<tr>
<td>3</td>
<td>1.920 x 10^6</td>
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<td>4</td>
<td>6.970 x 10^7</td>
<td>3.473 x 10^8</td>
</tr>
<tr>
<td>5</td>
<td>5.407 x 10^8</td>
<td>9.671 x 10^9</td>
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<td>6</td>
<td>9.021 x 10^9</td>
<td>5.798 x 10^10</td>
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<td>7</td>
<td>3.241 x 10^10</td>
<td>7.486 x 10^11</td>
</tr>
<tr>
<td>8</td>
<td>2.507 x 10^11</td>
<td>2.080 x 10^12</td>
</tr>
<tr>
<td>9</td>
<td>4.164 x 10^12</td>
<td>1.238 x 10^14</td>
</tr>
<tr>
<td>10</td>
<td>1.466</td>
<td>15.447</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>0.0036</td>
</tr>
</tbody>
</table>
it was possible to determine the amplitude $A = 0.82$ and the period of oscillation $T_{120} = 5.60s$ of the output signal, and also the magnitude of the FOI at the frequency $\omega_{120}$: $|F(j\omega_{120})| = 0.99$, completing the set of data which must be used for definition of PI/PID controller gains. The resulting set of PI/PID controller parameters is $K_p = 0.75$, $T_i = 5.04$ and $T_d = 0.16$. The closed-loop performance of the aircraft pitch angle with the inclusion of the PI/PID controllers is presented in Figure 5(b). The settling time ($t_s$) and the maximum overshoot ($M_o$) are about the same for both controllers - $t_s \approx 40$ s and $M_o \approx 12\%$ - with smaller oscillations with the PID controller, due to the larger phase margin.

The second plant to be considered for the application of the proposed PI/PID tuning method is the read/write head positioning system of a hard disk drive, presented in [5], which is described by the transfer function presented in equation (20).

$$G(s) = \frac{5.00 \times 10^5 s^2 + 1.40 \times 10^7 s + 4.67 \times 10^{10}}{s^4 + 178 s^3 + 1.09 \times 10^9 s^2 + 4.21 \times 10^7 s + 4.67 \times 10^{15}}$$

(20)

Although this is a self regulated plant with a zero frequency gain equal to one, its reaction curve, presented in figure 7(a), is not a well defined S-curve, excluding the use of reaction curve methods. The Nyquist diagram of $G(s)/2$ is shown in Figure 6, in green dashed line. It is seen that this graph does not cross the imaginary axis, thus excluding the application of the CFO method. But the Nyquist plot does cross the $-120^\circ$ line, represented with a full blue line, and this intersection corresponds to the oscillation that will be observed in the relay feedback experiment with the FOI.

Indeed, the relay feedback experiment with a FOI in the loop results in the output signal presented in figure 8(a), with period of oscillation $T = 0.027s$ and amplitude $A = 2.54$; the magnitude of the FOI at the frequency $\omega_{120}$ is $|F(j\omega_{120})| = 0.0282$. The resulting tuning of the PI/PID controller is $K_p = 0.41$, $T_i = 0.024$ and $T_d = 7.6 \times 10^{-4}$, and the Nyquist diagram of the loop transfer function $C(s)G(s)$ with the PID controller is also shown in Figure 6, as a full red line. The closed-loop responses to a unit step with both the PI and the PID controllers in the closed loop are presented in Figure 8(b). The settling time and the maximum overshoot are about the same for both controllers - $t_s \approx 0.4$ s and $M_o = 0$.

The two examples explored in this Section have very different dynamic characteristics, with the aircraft pitch angle

\[3\text{This change of scale was made to fit two plots in the same graph}\]
system presenting a much slower behavior when compared to the read/write head positioning system. Still, the same Fractional Order Integrator, with the same coefficients presented in the Table II, was successfully employed for both plants.

VI. A TEST BATCH

The proposed method was applied for the tuning of PI and PID controllers in five different classes of plants, with the controller tuning as presented in Table I. Each of the following Subsection details the application to one of these classes. The performance is assessed in each case by the settling time ($t_s$) and the maximum overshoot ($M_o$) in the closed-loop response to a step reference. The tuning method proposed in this paper has been developed for plants whose Nyquist plot does not cross the negative real axis, and the first four classes of plants considered in the following possess this feature. Yet, the method can also be applied to plants that do not fit into this category, and this is illustrated in Subsection VI-D where it is applied to plants with a transfer function in the first-order plus time delay form.

A. Second order plants with real poles

Consider the following class of plants:

$$G(s) = \frac{\alpha}{(s + 1)(s + \alpha)}$$

(21)

We have applied the method for thirteen different values of $\alpha$, from $\alpha = 10^{-2}$ up to $\alpha = 10^2$. The results are shown in Table III, with the open and closed loop step responses for $\alpha = 10^{-2}$ presented in Figure 9. It is seen that tuning by the EFO yields good performance for all values of $\alpha$. It is also seen that the PID controller typically provides about the same settling time as the PI controller, with smaller overshoot. This same pattern has been observed in all classes of plants to be presented next. So, in order to save space, only the results obtained with PI are presented in the sequel.

B. Second order plants with complex poles

Consider the following class of plants:

$$G(s) = \frac{1}{s^2 + 2\alpha s + 1}$$

(22)

We have applied the method for nine different values of $\alpha$ between 0 and 1. The results are shown in Table IV, with the
C. Third order plants with real poles

1) A lead block: Consider the following class of plants:

\[ G(s) = \frac{\alpha \left( \frac{s}{\sqrt{\alpha}} + 1 \right)}{(s + 1)(s + \alpha)(\frac{s}{\sqrt{\alpha}} + 1)} \]  \hspace{1cm} (23)

which consists of the second-order transfer function (21) with the addition of a lead block. The lead block in the transfer function (23) consists of a pole-zero pair in between the original poles at \(-1\) and \(-\alpha\), with \(\alpha\) as the product of its singularities. Again, we have applied the method for thirteen different values of \(\alpha\), from \(\alpha = 10^{-2}\) up to \(\alpha = 10^2\). The results are shown in Table V with the closed loop step response, including the PI controller with the gains tuned through the proposed methodology, presented in Figure 11.

2) A lag block: Consider the following class of plants:

\[ G(s) = \frac{\alpha \left( \frac{s^2}{\sqrt{\alpha}} + 1 \right)}{(s + 1)(s + \alpha)(\frac{s^2}{\sqrt{\alpha}} + 1)} \]  \hspace{1cm} (24)

which consists of the second-order transfer function (21) with the addition of a lag block. Once more, we have applied the method for thirteen different values of \(\alpha\), from \(\alpha = 10^{-2}\) up to \(\alpha = 10^2\), and the results are also shown in Table V. 4

Step responses for this and the next class of plants are not shown for lack of space.
D. FOPTD

The tuning method proposed in this paper is devised for plants whose Nyquist diagram does not cross the negative real axis. Yet, it can be also applied to plants whose frequency response does cross the negative real axis, with satisfactory results. In order to illustrate this application, consider the first order plus time delay (FOPTD) model:

\[
G(s) = \frac{e^{-\tau s}}{Ts + 1}
\]  

(25)

The Nyquist plot of this plant crosses the negative real axis for any positive values of \( \tau \) and \( T \). However, for very small delay - that is, \( \tau \to 0 \) - this crossing occurs at very high frequency and with a high slope. The CFO method relies on the frequency response being sufficiently smooth around the negative real axis, so for very small \( \tau \) ratios it does not provide good results (as will be illustrated numerically shortly). In such cases, because the Nyquist plot barely crosses the negative real axis, our method is expected to provide results similar to the ones obtained in plants without ultimate frequency. On the other hand, for moderate and large \( \tau \) ratios the CFO method is in its element, so we can expect it to perform better than our method.

In order to verify the above rationale, we tested the two methods - CFO and EFO - for the class of plants (25) with \( T = 1 \) and a range of values for the time delay \( \tau \). The results are shown in Table VI. It is seen that, as expected, our method provides acceptable performance for this class of systems, even outperforming the CFO for small values of the ratio \( \tau/T \).

VII. CONCLUSIONS

In this paper an alternative methodology was proposed - and baptized the EFO method - for tuning PID controllers for a class of plants which are not amenable to application of the relay feedback Ziegler-Nichols method - the so-called CFO. The EFO method extends the CFO method, thus serving as an alternative to the reaction curve method for this class of plants, but also applies to other classes of plants. The methodology is based on the inclusion in the loop of a transfer function with constant phase in a large range of frequencies, which can be achieved with a Fractional Order Integrator. The proposed method was validated considering five different classes of plant’s transfer functions, taking into account a set of different parameters for each class of plants, besides a detailed description of the application to two case studies. The closed-loop performance is similar to what is achieved with Ziegler-Nichols-like methods in general and the method is applicable to a large range of plants that do not admit the application of either classical Ziegler-Nichols tuning.

REFERENCES


\[
\text{FOPTD - PI}
\]

TABLE VI

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<tr>
<th>( \tau )</th>
<th>( K_p )</th>
<th>( T_i )</th>
<th>( T_d )</th>
<th>( M_o )</th>
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\[\tau = 1\]

\[\frac{2}{T}\]

\[\tau = \frac{2}{T}\]

\[\text{Fig. 11. Closed loop response for the plant described by the transfer function (23) with } \alpha = 2\]