2021 60th IEEE Conference on Decision and Control (CDC)
December 13-15, 2021. Austin, Texas

An optimal regularized instrumental variable method for errors-in-variables identification

Emerson C. Boeira[1] and Diego Eckhard[2]

Abstract— **This work addresses the design of theoretical optimal regularization matrices for the instrumental variable method in the errors-in-variables identification framework. The design is based on the asymptotic statistical properties of the regularized instrumental variable estimator and it provides new bounds for the use of regularization in this scenario as well as new ideas to parametrize and estimate the regularization matrix in practical situations. A numerical example shows the effectiveness of the optimal estimator in comparison with the classic least-squares and instrumental variable methods.**

I. INTRODUCTION

Over the last decade, the regularization feature has been widely researched and developed in the system identification scientific community [1]. This sudden growth of popularity is a direct consequence of the new and enlightening ideas exposed in recent works [2], [3], [4], [5], that connected machine learning concepts to the use and the estimation of an *optimal regularization* matrix in impulse response identification problems.

In this new paradigm, it has been demonstrated that the regularized identification technique was able to outperform the classical prediction error methods (PEM) in some scenarios, essentially because it can reduce the variance error at the cost of introducing a small bias. So, with this whole new interest and motivation in regularization research, this work approaches the use of this feature in a distinct category of problems: the identification of errors-in-variables (EIV) dynamic systems.

The identification of errors-in-variables dynamic systems is a wide topic in system identification literature, with several methods, applications, and analysis available, as it can be seen in [6], [7]. However, the main interest in this paper relies on the so-called *elementary* methods to identify the impulse response, such as the *least-squares* (LS) and the *instrumental variable* (IV) methods [7], along with the novel regularized extension of the IV, which is the *regularized instrumental variable* (RIV) method.

It's important to say that the RIV estimator for errors-in-variables identification was already been addressed in the context of data-driven control (DDC) in [8], [9]. In both these works, however, the use of the regularization feature was based on a data-driven *bayesian* perspective of the controller's identification problem. On the other hand, this work proposes a theoretical optimization of an estimation quality criterion using the regularization matrix as the optimization variable in an unconstrained scenario and in a specific constrained scenario.

*This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001, by the Conselho Nacional de Desenvolvimento Científico e Tecnológico - Brasil (CNPq), and by PETROBRAS S/A.

[1]Emerson C. Boeira is with the Programa de Pós-Graduação em Engenharia Elétrica, Universidade Federal do Rio Grande do Sul, Av. Osvaldo Aranha 103, Porto Alegre, Brazil, emerson.boeira@ufrgs.br

[2]Diego Eckhard is with the Departamento de Matemática Pura e Aplicada, Universidade Federal do Rio Grande do Sul, Av. Bento Gonçalves 9500, Porto Alegre, Brazil, diegoeck@ufrgs.br

II. TRUE SYSTEM DESCRIPTION AND SIGNALS PRELIMINARIES

In this paper, the true system considered for identification that generates the data is a single-input single-output linear stable system with the following mathematical description:

$$\begin{cases} y(t) = G_0(q)u_0(t) \\ u(t) = u_0(t) + \tilde{u}(t), \end{cases} \tag{1}$$

where q is the forward shift operator and $G_0(q)$ is the process' transfer function, that can be expanded as

$$G_0(q) = \sum_{k=0}^{\infty} g_0(k)q^{-k}, \tag{2}$$

with $g_0(k)$ representing the k-th coefficient of its impulse response. The signals $y(t) \in \mathbb{R}$ and $u(t) \in \mathbb{R}$ represent the system's measured output and input signals, respectively, while $u_0(t) \in \mathbb{R}$ is the input that actually excites the process and $\tilde{u}(t) \in \mathbb{R}$ represents the input noise that contaminates the measure. Both $u_0(t)$ and $\tilde{u}(t)$ are considered to be quasi-stationary signals, in the same sense as introduced and explored in [10], with $\tilde{u}(t)$ also being considered a Gaussian distributed white noise with zero mean and variance denoted by $\sigma_{\tilde{u}}^2$. Furthermore, both signals are presumed to be uncorrelated, i.e.: $\bar{E}\big[u_0(t+\tau)\tilde{u}(t)\big] = 0,\ \forall\tau$, where $\bar{E}\big[\cdot\big]$ is defined as in [10]: $\bar{E}\big[x(t)\big] \triangleq \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} E\big[x(t)\big]$, and it's an important operator in the analysis of quasi-stationary signals, while $E\big[\cdot\big]$ denotes the traditional expected value operator.

It's worth mentioning that the system addressed in this work can be seen as a particular case of an EIV dynamic system, with the main difference that in (1) the output noise is absent, while in a typical errors-in-variables problem, the effect of the output noise is usually taken into consideration. Additionally, the main motivation to study a system as (1) is that it can represent some data-driven control problems, as the Virtual Reference Feedback Tuning (VRFT) [11] with open-loop data, which is a very popular and disseminated DDC method.

III. SYSTEM MODEL AND ELEMENTARY METHODS

Since the main purpose of this work is to identify the impulse response of $G_0(q)$, the process model can be defined as a *Finite Impulse Response* (FIR) model:

$$G(q,\theta) = \sum_{k=0}^{n} g(k) q^{-k}, \tag{3}$$

where $g(k)$ represents the impulse response coefficients. Certainly, this is the simplest approach to estimate $G_0(q)$ [4], where the expansion demonstrated in (2) is truncated at a finite number of coefficients. In this scenario, it's possible to notice that if $G_0(q)$ describes a stable system, then its impulse response decays to zero. So, an n-th order FIR model can comprehend a good approximation to it.

From the model category chosen to describe the process, a prediction of the output can be achieved with linear regression:

$$\hat{y}(t) = \varphi(t)^T \theta, \tag{4}$$

where $\hat{y}(t)$ denotes the predicted output, $\varphi(t) \in \mathbb{R}^n$ denotes the regressor vector, and $\theta \in \mathbb{R}^n$ denotes the parameter vector, which represent the system's impulse response coefficients:

$$\varphi(t) = \begin{bmatrix} u(t) & u(t-1) & u(t-2) & \dots & u(t-n) \end{bmatrix}^T, \tag{5}$$

$$\theta = \begin{bmatrix} g(0) & g(1) & g(2) & \dots & g(n) \end{bmatrix}^T. \tag{6}$$

Finally, a very important assumption that is presumed in this paper and in most of the system identification literature to simplify the statistical and quality analysis of the estimators is stated in the sequence.

Assumption 1: The true process $G_0(q)$ belongs to the model structure $G(q,\theta)$, i.e. $\exists \theta_0 : G(q,\theta_0) = G_0(q)$.

When Assumption 1 is satisfied, it's possible to write the true system as a linear regression as well: $y(t) = \varphi_0(t)^T \theta_0$, where

$$\varphi_0(t) = \begin{bmatrix} u_0(t) & u_0(t-1) & \dots & u_0(t-n) \end{bmatrix}^T, \tag{7}$$

$$\theta_0 = \begin{bmatrix} g_0(0) & g_0(1) & \dots & g_0(n) \end{bmatrix}^T. \tag{8}$$

Also, since the signal $u_0(t)$ is unknown, the system's output can be described, alternatively, as a function of the measured output $u(t)$ and the input noise $\tilde{u}(t)$ [7]:

$$y(t) = \varphi(t)^T \theta_0 + \mu(t), \tag{9}$$

with $\mu(t) = -\tilde{\varphi}(t)^T \theta_0$ and where $\varphi(t)$ can be expressed as $\varphi(t) = \varphi_0(t) + \tilde{\varphi}(t)$, with

$$\tilde{\varphi}(t) = \begin{bmatrix} \tilde{u}(t) & \tilde{u}(t-1) & \dots & \tilde{u}(t-n) \end{bmatrix}^T. \tag{10}$$

These alternative expressions and quantities, such as $\mu(t)$ and $\tilde{\varphi}(t)$ are essential in the sequence to derive and understand which elements compose the statistical properties of the estimators that are addressed in this paper.

In order to identify the linear regression model presented in (4), using the measured input and output data $Z^N = \big[u(1), y(1), \dots, u(N), y(N)\big]$, this paper discusses two elementary methods alongside its properties in the sequence: the ordinary least-squares and the basic instrumental variable, where the latter is extended with the use of regularization.

A. Ordinary least-squares identification

The ordinary least-squares estimate of the parameter vector θ is defined as the minimizing argument of the sum of squared equation errors between the model prediction and the measured output [6], [7]:

$$\hat{\theta}_{ls} = \arg\min_{\theta} \frac{1}{N} \sum_{t=1}^{N} \left(y(t) - \varphi(t)^T \theta \right)^2, \tag{11}$$

which leads to the following normal equation [7]:

$$\hat{\theta}_{ls} = \left[\frac{1}{N} \sum_{t=1}^{N} \varphi(t) \varphi(t)^T \right]^{-1} \left[\frac{1}{N} \sum_{t=1}^{N} \varphi(t) y(t) \right]. \tag{12}$$

Despite the advantage of being a simple, well-known, and computationally efficient method, the least-squares identification possesses a major drawback in the errors-in-variables scenario: it produces estimates that are *not consistent* [7], i.e., that are biased even when $N \to \infty$. This characteristic can be undesirable in some system identification problems.

B. Basic instrumental variable identification

To overcome the bias error exposed above, a more advanced method can be applied to identify the model (4): the basic instrumental variable, which can be interpreted as a generalization of the least-squares [7]. The main idea of the IV method is to rewrite the normal equation as

$$\hat{\theta}_{iv} = \left[\frac{1}{N} \sum_{t=1}^{N} \zeta(t) \varphi(t)^T \right]^{-1} \left[\frac{1}{N} \sum_{t=1}^{N} \zeta(t) y(t) \right], \tag{13}$$

where $\zeta(t) \in \mathbb{R}^n$ is known as the instrumental variable vector, which must satisfy:

$$\bar{E}\big[\zeta(t)\varphi(t)^T\big] \text{ is invertible}, \quad \bar{E}\big[\zeta(t)\mu(t)\big] = 0, \tag{14}$$

to guarantee that the IV estimate is consistent, or unbiased in the limit $N \to \infty$ [7].

The instrumental variable vector can be built in several distinct ways, which can hold different results and different statistical properties, as discussed in more depth in [7]. However, this work focus on the instrumental variable produced with data collected through a second experiment on the system, which holds a distinct noise realization of $\tilde{u}(t)$:

$$\zeta(t) = \begin{bmatrix} u_2(t) & u_2(t-1) & \dots & u_2(t-n) \end{bmatrix}^T, \tag{15}$$

where $u_2(t) \in \mathbb{R}$ denotes the input signal obtained in the second experiment: $u_2(t) = u_0(t) + \tilde{u}_2(t)$, and $\tilde{u}_2(t)$ is the different noise realization that is uncorrelated with $\tilde{u}(t)$, i.e. $E\big[\tilde{u}(t+\tau)\tilde{u}_2(t)\big] = 0, \ \forall \tau$.

Yet, a great disadvantage of the IV identification method is the large *variance error* produced by its estimates when the number of available data is limited. Accordingly, to manage the *bias-variance trade-off* between both elementary methods (LS and IV), this work proposes an optimal regularized IV method, based on the innovative ideas that have been arising in the system identification community for impulse response identification [4], [5], [1].

IV. THE REGULARIZED INSTRUMENTAL VARIABLE ESTIMATOR

The use of the regularized instrumental variable estimator was already been applied in the data-driven control literature, specifically in the VRFT and CbT methods [8], [9]. In such context, it has been demonstrated that the regularized instrumental variable estimate is given by

$$\hat{\theta}_{riv} = \left[\frac{1}{N}\sum_{t=1}^{N} P\zeta(t)\varphi(t) + I\right]^{-1}\left[\frac{1}{N}\sum_{t=1}^{N} P\zeta(t)y(t)\right], \tag{16}$$

where $P \in \mathbb{R}^{n\times n}$ is known as the *regularization matrix*. The estimate produced by (16) can be interpreted as a natural generalization of the regularized least-squares methods that are used in state-of-the-art techniques in impulse response identification [4], [5], [1], in the same sense as the basic IV is a generalization of the ordinary LS.

Finally, to improve the statistical performance of the estimator (16) by adjusting the bias-covariance trade-off, the matrix P must be chosen properly. For such purpose, in [8], [9], for example, a Bayesian perspective, based on process data, was introduced for the CbT and VRFT techniques resulting in better properties if compared to the elementary methods. Distinctly, the major contribution of this paper is to find an optimal regularization matrix based on the theoretical statistical properties of $\hat{\theta}_{riv}$ to provide new insights on which quantities that it depends on and how its structure can be parametrized in algorithms to compute and to estimate it.

V. ASYMPTOTIC PROPERTIES OF THE REGULARIZED INSTRUMENTAL VARIABLE ESTIMATOR

In order to obtain the conditions to calculate the optimal regularization matrix regarding an estimator quality measure or criterion, this section exposes the asymptotic statistical properties of the regularized instrumental variable estimator (16). These statistical analyses are based on the asymptotic data scenario, i.e., $N \to \infty$, since it's a very difficult task to compute parameter distributions for the general case, as stated in [10]. Thus, it's important to emphasize that such properties can be considered approximations for the finite data scenario, which are more trustworthy as N increases.

The quality criterion chosen to be minimized in this work with the use of regularization is the *trace* of the MSE matrix since this a wide measure for the estimator quality that comprehends both the bias and the variance errors. Also, it's worth mentioning that a similar approach, that inspired this idea, was performed in the innovative state-of-the-art papers [4], [5], regarding impulse response estimation.

So, the definition of the MSE matrix of the RIV estimator can be expressed by $\mathcal{Q}(P) \triangleq E\Big[\big(\hat{\theta}_{riv} - \theta_0\big)\big(\hat{\theta}_{riv} - \theta_0\big)^T\Big]$. Also, this matrix can be associated with the bias and the covariance errors from: $\mathcal{Q}(P) = \mathcal{B}(P)^T\mathcal{B}(P) + \mathcal{V}(P)$, where it's relevant to recall that the bias and the covariance errors are defined by

$$\mathcal{B}(P) \triangleq E\big[\hat{\theta}_{riv}\big] - \theta_0, \tag{17}$$

$$\mathcal{V}(P) \triangleq E\Big[\big(\hat{\theta}_{riv} - E[\hat{\theta}_{riv}]\big)\big(\hat{\theta}_{riv} - E[\hat{\theta}_{riv}]\big)^T\Big]. \tag{18}$$

A. Asymptotic bias of the regularized instrumental variable estimator

Firstly, it's shown that the RIV estimator is indeed asymptotically biased, or not consistent and that the bias depends on the choice of the regularization matrix. So, to compute the bias, the first step is to consider the value θ^*_{riv}, for which the estimate converges when $N \to \infty$:

$$\theta^*_{riv} = \lim_{N\to\infty} \hat{\theta}_{riv} = \tag{19}$$

$$\lim_{N\to\infty}\left[\frac{1}{N}\sum_{t=1}^{N} P\zeta(t)\varphi(t)^T + I\right]^{-1}\left[\frac{1}{N}\sum_{t=1}^{N} P\zeta(t)y(t)\right]. \tag{20}$$

From (20) and from the fact that the signals considered in this work are all *quasi-stationary* and *ergodic*, the asymptotic sums can be substituted by the $\bar{E}[\,\cdot\,]$ operator [10]: $\theta^*_{riv} = \Big(P\bar{E}\big[\zeta(t)\varphi(t)^T\big] + I\Big)^{-1}\Big(P\bar{E}\big[\zeta(t)y(t)\big]\Big)$. Now, the relation between the instrumental variable, obtained in the second experiment, and the regressor vector, from the first experiment, can be observed: $\varphi(t) = \varphi_0(t) + \tilde{\varphi}(t)$, $\zeta(t) = \varphi_0(t) + \tilde{\varphi}_2(t)$, and since $\varphi_0(t)$, $\tilde{\varphi}(t)$ and $\tilde{\varphi}_2(t)$ are uncorrelated vectors, $\bar{E}\big[\zeta(t)\varphi(t)^T\big]$ simplifies to:

$$\bar{E}\big[\zeta(t)\varphi(t)^T\big] = \bar{E}\big[\varphi_0(t)\varphi_0(t)^T\big] = R_0. \tag{21}$$

Also, since $y(t) = \varphi_0(t)^T\theta_0$, then:

$$\bar{E}\big[\zeta(t)y(t)\big] = R_0\theta_0. \tag{22}$$

So, from (21) and (22) the value of $\hat{\theta}_{riv}$ can be calculated, as $N \to \infty$, through $\theta^*_{riv} = \big(PR_0 + I\big)^{-1}PR_0\theta_0$, which results in the following expression for the asymptotic bias of the regularized instrumental variable: $\lim_{N\to\infty}\mathcal{B}(P) = -\big(PR_0 + I\big)^{-1}\theta_0$, where the dependency on the regularization matrix becomes explicit.

B. Asymptotic Mean-Square Error matrix for the regularized instrumental variable estimator

Here, the theoretical asymptotic MSE matrix of the RIV estimator is briefly demonstrated to determine its dependency on P and consequently, to optimize its trace. The expression for such quantity can be obtained with the analysis on its definition and by using the auxiliary variable $\mathbf{X}_0$: $\lim_{N\to\infty}\mathcal{Q}(P) = E\big[\mathbf{X}_0\mathbf{X}_0^T\big]$, with $\mathbf{X}_0$ being defined as:

$$\mathbf{X}_0 = \lim_{N\to\infty}\hat{\theta}_{riv} - \theta_0, \tag{23}$$

$$= \lim_{N\to\infty}\left[\frac{1}{N}\sum_{t=1}^{N} P\zeta(t)\varphi(t)^T + I\right]^{-1}\left[\frac{1}{N}\sum_{t=1}^{N} P\zeta(t)\mu(t) - \theta_0\right], \tag{24}$$

recalling that, from (9), $\mu(t) = y(t) - \varphi(t)^T\theta_0 = G_0(q)\tilde{u}(t)$. With the expression above and with the *strong law of large numbers*, the following approximation can be performed:

$$\lim_{N\to\infty}\left[\frac{1}{N}\sum_{t=1}^{N}P\zeta(t)\mu(t)^T + I\right] \to \big(PR_0 + I\big), \quad (25)$$

which results in:

$$\mathbf{X}_0 = \lim_{N\to\infty}\big(PR_0 + I\big)^{-1}\left[\frac{1}{N}\sum_{t=1}^{N}P\zeta(t)\mu(t)^T - \theta_0\right]. \quad (26)$$

From such approximation, the asymptotic expression for $\mathcal{Q}(P)$ can be calculated through:

$$\lim_{N\to\infty}\mathcal{Q}(P) = \big(PR_0 + I\big)^{-1}E\big[\mathbf{X}_{in}\mathbf{X}_{in}^T\big]\big(R_0P^T + I\big)^{-1}, \quad (27)$$

with

$$\mathbf{X}_{in} = \lim_{N\to\infty}\left[\frac{1}{N}\sum_{t=1}^{N}P\zeta(t)\varphi(t)^T - \theta_0\right]. \quad (28)$$

Finally, to summarize, it can be observed that $E\big[\mathbf{X}_{in}\mathbf{X}_{in}^T\big]$ can be calculated as $E\big[\mathbf{X}_{in}\mathbf{X}_{in}^T\big] = PWP^T + \theta_0\theta_0^T$, where

$$W = \lim_{N\to\infty}E\left[\frac{1}{N^2}\sum_{t=1}^{N}\sum_{s=1}^{N}\zeta(t)\mu(t)\mu(s)\zeta(t)^T\right], \quad (29)$$

and it's possible to demonstrate that W can be computed as [7]

$$W = \lim_{N\to\infty}\frac{\sigma_{\tilde{u}}^2}{N}\Bigg\{\bar{E}\left[\big(G_0(q)\varphi_0(t)\big)\big(G_0(q)\varphi_0(t)\big)^T\right] + \bar{E}\left[\big(G_0(q)\tilde{\varphi}_2(t)\big)\big(G_0(q)\tilde{\varphi}_2(t)\big)^T\right]\Bigg\}, \quad (30)$$

which leads to the final expression for the MSE matrix of the $\hat{\theta}_{riv}$ estimator:

$$\lim_{N\to\infty}\mathcal{Q}(P) = \big(PR_0 + I\big)^{-1}\big(PWP^T + \theta_0\theta_0^T\big)\big(R_0P^T + I\big)^{-1}. \quad (31)$$

Additionally, considering that the expression for $\mathcal{Q}(P)$ is legitimate for the asymptotic MSE scenario, it can be approximated for a finite amount of data by the following expression: $\bar{\mathcal{Q}}(P) = \big(PR_0 + I\big)^{-1}\big(P\bar{W}P^T + \theta_0\theta_0^T\big)\big(R_0P^T + I\big)^{-1}$, with

$$\bar{W} = \frac{\sigma_{\tilde{u}}^2}{N}\Bigg\{\bar{E}\left[\big(G_0(q)\varphi_0(t)\big)\big(G_0(q)\varphi_0(t)\big)^T\right] + \bar{E}\left[\big(G_0(q)\tilde{\varphi}_2(t)\big)\big(G_0(q)\tilde{\varphi}_2(t)\big)^T\right]\Bigg\}. \quad (32)$$

VI. THE OPTIMAL REGULARIZATION MATRIX

As mentioned before, this work seeks the regularization matrix that optimizes the trace of the MSE matrix, which can be formulated as the following mathematical problem:

$$P_0 = \arg\min_{P} tr\big[\bar{\mathcal{Q}}(P)\big], \quad (33)$$

where the solution is obtained by computing the derivative of $tr\big[\bar{\mathcal{Q}}(P)\big]$ with relation to P and solving the following matrix equation:

$$\frac{\partial tr\big[\bar{\mathcal{Q}}(P)\big]}{\partial P} = 0. \quad (34)$$

From matrix calculus and the following matrix identities:

$$\begin{cases}\big(R_0P^T + I\big)^{-T} = \big(PR_0 + I\big)^{-1}, \\ \big(PR_0 + I\big)^{-T} = \big(R_0P^T + I\big)^{-1},\end{cases} \quad (35)$$

the derivative exposed in (34) has a certain degree of symmetry and results in

$$R_0\big(PR_0 + I\big)^{-1}\big(\theta_0\theta_0^T + P\bar{W}P^T\big) - \bar{W}P^T = 0. \quad (36)$$

Furthermore, after some algebraic manipulation, it can be noticed that (36) has the following unique solution:

$$P_0 = \theta_0\theta_0^T R_0\bar{W}^{-1}. \quad (37)$$

Now, from the theoretical expression exposed in (37), some conclusions about P_0's structure and properties can be drawn. Initially, it's fundamental to realize that P_0 depends on unknown quantities, as θ_0, R_0 and $\bar{W}$, and so, in practical situations, it can't be computed and used directly. Still, it can be noticed that P is such that $\theta^T P\theta > 0,\ \forall\theta \in \mathbb{R}$.

The expression (37) and the characteristics highlighted above are relevant because they provide new insights and new ideas for possible parametrization structures, for example, in novel algorithms to estimate P_0 in practical identification problems.

Also, if desired by the user, some restrictions can be imposed in the optimization problem (33), such as the symmetry of the regularization matrix, for example, which reduces the number of free parameters. In this context, the problem can be rewritten as

$$\begin{cases}P_{0s} = \arg\min_{P} tr\big[\mathcal{Q}(P)\big], \\ s.t.\ P = P^T.\end{cases} \quad (38)$$

In (38), however, the aforementioned symmetry in the derivatives is not present. So, the matrix identities (35) can't be applied to simplify the problem, leading to a more complex matrix equation:

$$\big(\theta_0\theta_0^T R_0R_0 - W\big)P + P\big(R_0R_0\theta_0\theta_0^T - W\big) - P\big(WR_0 + R_0W\big)P + \big(R_0\theta_0\theta_0^T + \theta_0\theta_0^T R_0\big) = 0. \quad (39)$$

that can be found after some of degree of algebraic manipulation, that will not be presented here due to the lack of space to develop such calculations. The equation (39) is actually a Riccati equation, which is very popular in control systems literature and possesses a wide range of solvers available to compute its solution.

VII. NUMERICAL EXAMPLE

This section elaborates a numerical example to show the efficiency of the optimal RIV method in comparison to the elementary ones under the context of errors-in-variables impulse response estimation. Nonetheless, it's essential to recall that these results can just be interpreted as higher or lower bounds, since P_0 and P_{0s} are based on theoretical solutions and unknown quantities for (33) and (38). The estimation of these matrices and their comparison with the optimal case in practical identification situations will be addressed in future works.

The process considered for estimation in this example has the same structure as described in (1), where $G_0(q)$ is an FIR filter with $n = 35$ coefficients, that represents the truncated impulse response of the following Butterworth filter:

$$G_{iir}(q) = \frac{0.02008q^2 + 0.04017q + 0.02008}{q^2 - 1.561q + 0.6414}. \quad (40)$$

The same system was also used as an example on [1]. The input signal $u_0(t)$ applied to excite the system was a square wave that oscillates between $[0, 1]$, with period $T_{sq} = 100\ samples$, and the noise signal $\tilde{u}(t)$ generated to corrupt the measure was chosen as a Gaussian distributed white noise with zero mean and variance $\sigma^2_{\tilde{u}} = 0.025$. These choices result in a signal-to-noise ratio of 10.

The system was simulated through 1000 Monte Carlo runs and $N = 2500$ data samples were collected in each run to perform the identification with the elementary methods and the RIV with both P_0 and P_{0s}. To compare all methods, some metrics were evaluated, where the first one concerns the statistical properties produced by each one. Regarding these results, Table I demonstrates the bias norm, the trace of the covariance matrix and the trace of the MSE matrix that were obtained in the simulations.

TABLE I

COMPARISON BETWEEN BIAS VECTOR NORM, TRACE OF THE COVARIANCE MATRIX AND TRACE OF THE MSE MATRIX PRODUCED BY EACH IDENTIFICATION METHOD.

	$\|\|\mathcal{B}\|\|_2$	$tr(\mathcal{V})$	$tr(\mathcal{Q})$
LS	4.1994×10^{-2}	1.2347×10^{-4}	1.8893×10^{-3}
IV	2.6787×10^{-3}	3.1117×10^{-3}	3.1149×10^{-3}
RIV+P_0	6.3279×10^{-5}	2.7717×10^{-6}	2.7758×10^{-6}
RIV+P_{0s}	6.5161×10^{-5}	2.7954×10^{-6}	2.7996×10^{-6}

Table I shows that, with the use of the optimal regularization matrices, the RIV method yields the best results for the trace of the MSE, where the difference between using P_0 and P_{0s} was relatively small. The same table also shows that the norm of the bias vector was larger for the least-squares method (which was an expected result) and that for the regularized ones, such norm was significantly small. It's worth mentioning, that the bias achieved by the IV estimator was quite large due to the limited number of data samples, Monte Carlo runs, and its large covariance error. Regarding the trace of the covariance matrix, Table I shows that the IV estimates held the larger value for this quantity (also an expected result), and with the use of the optimal regularization, this property has been considerably enhanced in the RIV technique, as desired.

Another interesting analysis that was performed in this work involves a comparison between the impulse responses that were estimated in each Monte Carlo run. This comparison provides a good visual idea of what was accomplished by each method. Within this purpose, Fig. 1 exhibits all the 1000 estimated impulse responses for the ordinary least-squares and basic instrumental variable methods and Fig. 2 exhibits the same graphics for the regularized instrumental variable estimator with P_0 and P_{0s} respectively. These graphics demonstrate the effect of the bias error on the LS estimated impulse responses as long as the effect of the large variance error for the IV estimated responses and the improvement with the use of the optimal RIV.

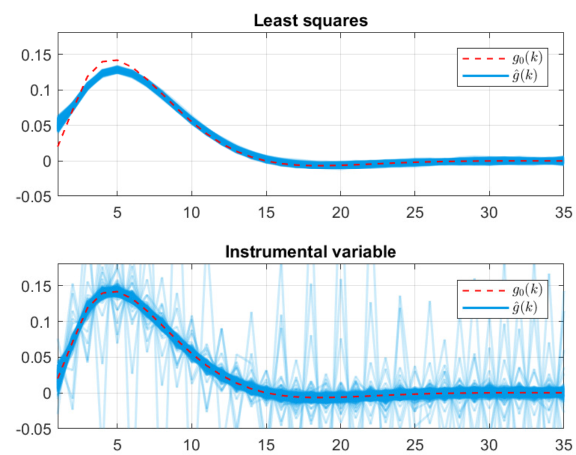

Fig. 1. Comparison of the estimated impulse responses with the elementary methods: least-squares and instrumental variable. Blue solid lines represent each one of the estimated impulse responses and the red dashed line represents the true impulse response of the process.

Finally, the last criterion to compare the results achieved in this work is the distribution of a fit measure of the impulse response estimations. The same measure was also applied in the classical regularization works [4], [5] and it consists in

$$F = \left[1 - \left(\frac{\sum_{k=0}^{n} |g_0(k) - \hat{g}(k)|^2}{\sum_{k=0}^{n} |g_0(k) - \bar{g}_0(k)|^2} \right)^{1/2} \right], \quad (41)$$

with $\bar{g}_0 = \frac{1}{n}\sum_{k=0}^{n} g_0(k)$, where $F \approx 1$ means a good fit between the estimated and the true impulse response. The boxplot of this measure is exhibited, for each method, in Fig. 3.

Fig. 3 demonstrates that the LS method produced a fit distribution with a smaller median in comparison with the others, which is a direct consequence of the method's bias error. Fig. 3 also demonstrates that the IV method produced a fit distribution with a higher median, closest to one, but with larger variance and with several outliers, which isn't a very good result as well. On the other hand, the boxplot graphic shows that the fit measure for the optimal RIV methods

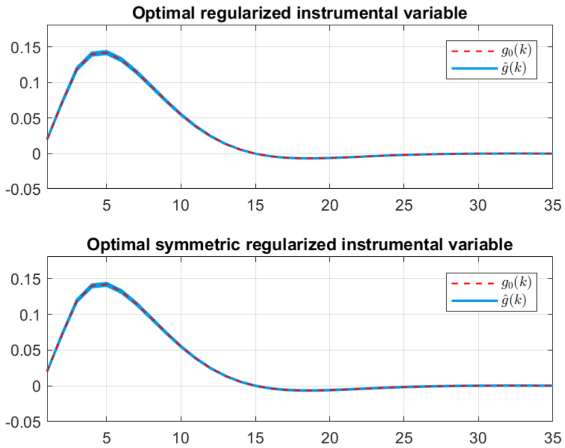

Fig. 2. Comparison of the estimated impulse responses with the regularized instrumental variable method: optimal and symmetrical optimal regularization matrices. Blue solid lines represent each one of the estimated impulse responses and the red dashed line represents the true impulse response of the process.

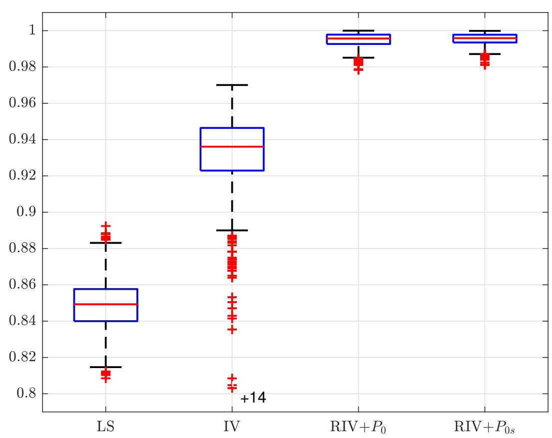

Fig. 3. Boxplot of the 1000 fits obtained on the Monte Carlo runs for each errors-in-variables identification method.

achieved a median very close to 1 with a small variance as well, which is a very good result.

To conclude the analysis, the numerical example confirms, through a series of comparisons and analysis, that the optimal RIV method produce better impulse response estimation if compared to the elementary methods regarding several distinct criteria. In addition, the use of the P_0 and P_{0s} demonstrates an optimal bound on the use of regularization for the IV method in this context, as they can't be used in practical identification scenarios for depending on unknown quantities.

VIII. Concluding Remarks

In conclusion, this work has demonstrated the theoretical expression for the unconstrained optimal regularization matrix and the Riccati equation that produces the optimal symmetric matrix for the estimator described in (16). The achieved results demonstrated that the optimal matrices depend on some knowledge about the process, which is actually unavailable for practical applications. However, the results provide a bound of the use of regularization along with useful and enlightening ideas regarding these matrices structures and their main characteristics. Such new insights could be used to further parametrize and estimate the optimal matrices in practical EIV identification problems.

The numerical example also demonstrated the effectiveness of the optimal regularization matrices comparing their use with the traditional and elementary EIV identification methods: IV and LS. The example exhibited that the optimal RIV produces better results in several criteria comparisons, as some statistical quantities based on the Monte Carlo simulations, the comparison of the impulse responses obtained by each method and the FIT error metric, which is widely employed in system identification literature.

Nevertheless, there are still some work and some research to be further explored and develop in this theme. One possible research topic, for example, is the extension of this work to a more general EIV system description with colored input and output noise. Another relevant idea to explore in this field may be the estimation of the regularization matrix from data, which is an essential matter for the practical application of the RIV method. This estimation could be performed through a *bayesian* interpretation of the problem, as in classical regularization works [4], [5], as well as some new and interesting *deep learning* algorithms, as the one exhibited in [12].

References

[1] L. Ljung, T. Chen, and B. Mu, "A shift in paradigm for system identification," *International Journal of Control*, vol. 93, no. 2, pp. 173–180, 2019.

[2] G. Pillonetto and G. De Nicolao, "A new kernel-based approach for linear system identification," *Automatica*, vol. 46, no. 1, pp. 81–93, 2010.

[3] G. Pillonetto, A. Chiuso, and G. De Nicolao, "Prediction error identification of linear systems: a nonparametric gaussian regression approach," *Automatica*, vol. 47, no. 2, pp. 291–305, 2011.

[4] T. Chen, H. Ohlsson, and L. Ljung, "On the estimation of transfer functions, regularizations and gaussian processes - revisited," *Automatica*, vol. 48, no. 8, pp. 1525–1535, 2012.

[5] G. Pillonetto, F. Dinuzzo, T. Chen, G. De Nicolao, and L. Ljung, "Kernel methods in system identification, machine learning and function estimation: a survey," *Automatica*, vol. 50, no. 3, pp. 657–682, 2014.

[6] T. Söderström, "Errors-in-variables methods in system identification," *Automatica*, vol. 43, no. 6, pp. 939 – 958, 2007.

[7] ——, *Errors-in-variables methods in system identification*. Amsterdam: Springer, 2018.

[8] S. Formentin and A. Karimi, "Enhancing statistical performance of data-driven controller tuning via l2-regularization," *Automatica*, vol. 50, no. 5, pp. 1514–1520, 2014.

[9] G. Rallo, S. Formentin, A. Chiuso, and S. M. Savaresi, "Virtual reference feedback tuning with bayesian regularization," in *Proceedings of the European Control Conference*. Aalborg: New York, IEEE, June 2016, pp. 507–512.

[10] L. Ljung, *System identification: theory for the user*, 2nd ed. Englewoods Cliff: Prentice Hall, 1999.

[11] M. C. Campi, A. Lecchini, and S. M. Savaresi, "Virtual reference feedback tuning: a direct method for the design of feedback controllers," *Automatica*, vol. 38, no. 8, pp. 1337–1346, 2002.

[12] C. Andersson, N. Wahlström, and T. B. Schön, "Data-driven impulse response regularization via deep learning," *IFAC-PapersOnLine*, vol. 51, no. 15, pp. 1–6, 2018, 18th IFAC Symposium on System Identification SYSID 2018.