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RESEARCH ARTICLE

Robust Convergence of the Steepest Descent Method for Data-Based Control

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Iterative data-based controller tuning consists of iterative adjustment of the controller parameters towards the parameter values which minimize an H_2 performance criterion. The convergence to the global minimum of the performance criterion depends on the initial controller parameters and on the step size of each iteration. This paper presents convergence properties of iterative algorithms when they are affected by disturbances.

Keywords: Robust Estimation, Discrete Time Systems, Stochastic Approximation, Nonlinear Programming.

1. Introduction

Several data-based control design methods utilize iterative algorithms to find the minimum of H_2 performance criteria, which express either one or a combination of the fundamental control objectives: reference tracking, noise rejection and economical use of control energy. These iterative data-based methods utilize data obtained from an experiment to compute estimates of the gradient of the cost function and maybe of its Hessian. These estimates are then used to feed an iterative optimization algorithm, such as the steepest descent or the Newton-Raphson method. Such methods as *Frequency Domain Tuning* (FDT) (Kammer et al. 2000), *Correlation based Tuning* (CbT) (Karimi et al. 2004, 2003) and *Iterative Feedback Tuning* (IFT) (Hjalmarsson et al. 1998, Gevers 2002) share this same formulation, which is also common to *Model Reference Adaptive Controllers* (MRAC) (Wang 1999). They differ from each other both in the particular H_2 performance criterion each one minimizes and in the particular means of obtaining appropriate estimates of the quantities necessary for the minimization.

The choice of the step size is critical to the performance of the algorithms (Hjalmarsson et al. 1998, Kammer et al. 2000, Karimi et al. 2004). The classical choice of the step size (Hjalmarsson et al. 1998, Hjalmarsson 2002) is robust to uncertainty on the estimates but only ensures local convergence and it has low convergence rate. More recent results on the choice of the step size can ensure convergence to the global minimum (when the initial condition is inside the domain of attraction) with high convergence rate but do not deal with disturbances on the estimates (Bazanella et al. 2008, Eckhard and Bazanella 2009, 2010). The aim of this work is to bridge this gap, by providing robustness guarantees for these faster step size choices.

The paper is organized as follows. Section 2 presents the definitions and the problem statement, and in Section 3 iterative algorithms are given. The step size policies are shown in Section 4. Section 5 presents the main result of this work. Simulations are presented in Section 6 and concluding remarks are given in Section 7.

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2. System definitions and notation

Consider a linear time-invariant discrete-time single-input single-output process

$$y(t) = G(q)u(t) + \nu(t), \tag{1}$$

where q is the forward time-shift operator, $G(q)$ is the process transfer function, assumed rational and proper, $u(t)$ is the control input and $\nu(t)$ is the process noise. The noise is a quasi-stationary process which can be written as $\nu(t) = H(q)e(t)$, where $e(t)$ is white noise with variance σ_e^2. This process is controlled by a linear time-invariant controller $C(q, \rho)$ which is assumed to have a parametric structure as specified below.

Assumption 2.1 Linear Parametrization

$$C(q, \rho) = \rho^T \bar{C}(q),$$

where $\rho \in \mathcal{D}_\rho \subseteq \mathrm{R}^p$ and $\bar{C}(q)$ is a column vector of fixed rational functions.

The control action $u(t)$ is given by

$$u(t) = C(q, \rho)(r(t) - y(t)), \tag{2}$$

where $r(t)$ is the reference signal, which is assumed to be quasi-stationary and uncorrelated with the noise. The closed-loop system (1)-(2) becomes

$$y(t, \rho) = T(q, \rho)r(t) + S(q, \rho)\nu(t)$$

$$T(q, \rho) \triangleq \frac{C(q, \rho)G(q)}{1 + C(q, \rho)G(q)} = C(q, \rho)G(q)S(q, \rho).$$

Let us define the set Γ of all control parameters values that render the closed-loop system BIBO-stable, that is, $\Gamma \triangleq \{\rho : T(q, \rho) \text{ is BIBO-stable}\}$.

We want the closed-loop to achieve a given performance which is specified by a "desired" closed-loop transfer function

$$y_d(t) = T_d(q)r(t),$$

called *the reference model*. We thus search for the controller parameters that make the output of the system the closest to the desired one, by solving the following optimization problem.

$$\min_\rho J(\rho)$$

$$J(\rho) \triangleq \bar{E}\left[((T(q, \rho) - T_d(q))r(t))^2\right]. \tag{3}$$

Once the desired closed-loop transfer function $T_d(q)$ is chosen, it could be exactly achieved with the ideal controller

$$C_d(q) = \frac{T_d(q)}{G(q)(1 - T_d(q))}. \tag{4}$$

This is the controller that minimizes the tracking error criterion $J(\rho)$. If and only if the ideal controller $C_d(q)$ lies within the class of controllers considered the closed-loop system can behave exactly as specified by the reference model. Let us formalize this assumption, which is referred as the Matched Control Case:

Assumption 2.2

$$\exists \rho_d \in \mathcal{D}_\rho : C(q, \rho_d) = C_d(q) = \rho_d^T \bar{C}(q).$$

We know from (Bazanella et al. 2008) that if Assumption 2.2 is ensured then the gradient can be written as

$$\nabla J(\rho) = \frac{\partial J(\rho)}{\partial \rho} = M(\rho)(\rho - \rho_d) \tag{5a}$$

$$M(\rho) = \frac{1}{\pi} \int_{-\pi}^{\pi} \Phi_r(e^{j\omega}) \left| G(e^{j\omega}) S(e^{j\omega}, \rho) \right|^2 \Re \left\{ S_d^*(e^{j\omega}) S(e^{j\omega}, \rho) \bar{C}(e^{j\omega}) \bar{C}^*(e^{j\omega}) \right\} d\omega \tag{5b}$$

where $\Re\{\cdot\}$ denotes the real part of a complex number.

3. The steepest descent algorithm

In adaptive control and data-based control design a model for the process is not known *a priori*, so neither is the cost function $J(\rho)$. Only local information about the cost function can be obtained from data collected on the system, so it is very common to use iterative gradient-based methods to find a minimum of the cost function. The steepest descent algorithm has the following structure

$$\rho_{i+1} = \rho_i - \gamma_i \nabla J(\rho_i) \tag{6}$$

where $\gamma_i > 0 \ \forall i$. The updates of the algorithm are made in the opposite direction of the gradient, so, at least for sufficiently small γ_i, at each iteration a smaller value for the cost is achieved.

Each data-based method utilizes a different manner to estimate the gradient of the cost function $\nabla J(\rho)$. The method *Frequency Domain Tuning* (FDT) (Kammer et al. 2000) uses a frequency domain approach to obtain the estimates considering zero reference. The method *Correlation based Tuning* (CbT) (Karimi et al. 2004) uses instrumental variables ideas. The method *Iterative Feedback Tuning* (IFT) (Hjalmarsson et al. 1998, Hjalmarsson 2002, Gevers 2002) uses data form two specific closed-loop experiments to generate an unbiased estimate of the cost function gradient. Whatever the procedure used to estimate the gradient, a stochastic error in this estimate exists. Robustness of the convergence properties with respect to this error is the concern of this paper.

A major concern in optimization, and particularly in the H_2 control design formulation, is the convergence to the global minimum of the performance criterion. This can usually not be achieved globally, that is, regardless of the initialization of the algorithm, so we define the following.

Definition 3.1 A set $\Omega \subset \mathrm{R}^p$ is a domain of attraction of an algorithm $\rho_{i+1} = f(\rho_i)$ for the function $J(\rho)$ if $\lim_{i\to\infty} \rho_i = \rho_d, \forall \rho_0 \in \Omega$.

When the steepest descent algorithm is used and the initial condition is sufficiently close to the global minimum, convergence to it depends only on the step size sequence γ_i. So, the problem of ensuring convergence of the steepest descent algorithm boils down to choosing appropriately the sequence of step sizes γ_i. However, this is not straightforward. If the step sizes are too large then the algorithm may leave the domain of attraction; if they are too small then the algorithm may need too many iterations to converge. In the next section two ways to compute the step size sequence will be shown.

4. Step size sequence

4.1 *Robins Monroe*

Several papers on data-based control give the following classical result as a theoretical foundation for the choice of the step size sequence.

Theorem 4.1 : *(Hjalmarsson et al. 1998, Hjalmarsson 2002) If the optimization problem is unconstrained, the estimate of the gradient is uniformly bounded and the step size γ_i of the steepest descent algorithm respects the following conditions:*

$$\sum_{i=1}^{\infty} \gamma_i = \infty \quad \sum_{i=1}^{\infty} \gamma_i^2 < \infty$$

then

$$\lim_{i \to \infty} \rho_i = \{\rho : \nabla J(\rho) = 0\}.$$

A "classical" choice of the step sizes, which satisfies these conditions, is the harmonic series (Hjalmarsson et al. 1998, Huusom et al. 2009):

$$\gamma_i = \frac{\gamma_1}{i}; \; i > 1. \tag{7}$$

This result shows that if the step size sequence respects (7) then the algorithm converges to an extremum of the cost function. The proof of the theorem is based on the stochastic convergence of the sequence ρ_i, so this step size sequence ensures convergence to an extremum even when the estimates of the gradient are affected by noise. This step sequence choice is theoretically justified, and as such is often presented as a benchmark, although it tends to provide poor convergence rates. Also, the choice of the step size of the first iteration is critical in the application of this policy.

4.2 *Proposed step size*

Some recent works present a way to design the step size sequence whose convergence rate seems to be the highest for which one can guarantee convergence in a general setting (Eckhard and Bazanella 2009). The following rule to choose the step size sequence is proposed

$$\gamma_i = \frac{\lambda_{min}(sym(M(\rho_i)))}{\|M^T(\rho_i)M(\rho_i)\|} \tag{8}$$

where $\lambda_{min}(\cdot)$ denotes the minimum eigenvalue and sym denotes the symmetrical part of a matrix $sym(X) = (X + X^T)/2$. It was shown in (Eckhard and Bazanella 2009) that this step size makes the algorithm converge to the minimum faster than the classical choice (7). However, this result was not developed considering the effects of disturbances on the estimates, so they do not ensure convergence when there is noise affecting the estimates. This work intends to show what happens to the steepest descent algorithm when the step size (8) is used and the estimates are disturbed. It will be shown that in this case the algorithm converges to a set whose size is proportional to the magnitude of the disturbance and that is centred at the global minimum.

5. Robustness

In this section it is analysed the convergence of the algorithm (6) when the gradient estimates are disturbed. The source of this disturbance most commonly comes from the estimates of the cost function gradient. Most the data-based methods present unbiased estimates to the cost function gradient, however these estimates have variance errors. In this section it will be considered that the step size sequence γ_i is designed to ensure convergence considering that there are no disturbances affecting the algorithm as presented in (Eckhard and Bazanella 2009). Then it will be shown that under mild assumptions on the disturbance the disturbed algorithm converges to a set that includes the global minimum.

Our first result shows that a discrete time system that has a minimum convergence rate, when disturbed converges to a set centred at the minimum and whose size is proportional to the disturbance magnitude. Then, it will be shown that the steepest descent algorithm has a minimum convergence rate when the step size (8) is used. It will be concluded that the disturbed steepest descent algorithm converges to a set centred at the global minimum when the step size policy (8) is used.

Lemma 5.1: *Consider a discrete time system*

$$x_{i+1} = f(x_i)$$

with equilibrium point x_* admitting a quadratic Lyapunov function

$$V(x_i) = (x_i - x_*)^T (x_i - x_*)$$

and that $\exists \beta < 1, \alpha > 0$ such that

$$V(f(x)) < \beta^2 V(x), \ \forall x \in \mathcal{B}_\alpha(x_*).$$

where

$$\mathcal{B}_\alpha(x_*) = \{x : \|(x - x_*)\| < \alpha\}.$$

Consider also that the system is perturbed by the disturbance g_i

$$x_{i+1} = f(x_i) + g_i \tag{9}$$

where $\|g_i\| < \delta \ \forall i$. Consider the set

$$\mathcal{S}(x_*) = \left\{ x_i : \|(x - x_*)\| < \frac{\delta}{1-\beta} \right\}.$$

and that $\mathcal{S}(x_*) \subset \mathcal{B}_\alpha(x_*)$. Then for all $x_0 \in \mathcal{B}_\alpha(x_*)$ the disturbed system (9) converges to the invariant set $\mathcal{S}(x_*)$.

Proof To study the convergence of the system, the quadratic Lyapunov function $V(x_i)$ is used, along with a candidate domain of attraction $\mathcal{B}_\alpha(x_*)$. For all $x_0 \in \mathcal{B}_\alpha(x_*)$ the disturbed system (9) converges to $\mathcal{S}(x_*)$ if

$$V(x_{i+1}) - V(x_i) < 0; \ \forall x_i \in \{\mathcal{B}_\alpha(x_*) - \mathcal{S}(x_*)\}. \tag{10}$$

Using the conditions of the theorem

$$\begin{aligned} V(x_{i+1}) - V(x_i) &= \|(f(x_i) - x_*)\|^2 - \|(x_i - x_*)\|^2 + 2g_i^T \|(f(x_i) - x_*)\| + g_i^T g_i \\ &< \beta^2 \|(x_i - x_*)\|^2 - \|(x_i - x_*)\|^2 + 2\delta\beta\|(x_i - x_*)\| + \delta^2 \; \forall x_i \in \mathcal{B}_\alpha(x_*). \quad (11) \end{aligned}$$

Then a sufficient condition to ensure (10) is

$$(\beta\|(x_i - x_*)\| + \delta)^2 - \|(x_i - x_*)\|^2 < 0 \; \forall x_i \in \{\mathcal{B}_\alpha(x_*) - \mathcal{S}(x_*)\}.$$

From the definition of $\mathcal{S}(x_*)$ it is easy to see that the above condition is always verified. It is still necessary to show that $\mathcal{S}(x_*)$ is an invariant set. From the assumptions of the theorem, if $x_i \in \mathcal{S}(x_*)$ then

$$\|x_{i+1} - x_*\| = \|f(x_i) - x_* + g_i\| < \|f(x_i) - x_*\| + \|g_i\| < \beta\|x_i - x_*\| + \|g_i\| < \frac{\beta\delta}{1-\beta} + \delta = \frac{\delta}{1-\beta}$$

which verifies that $\mathcal{S}(x_*)$ is invariant.

□

This theorem shows that the system disturbed by g_i converges to an invariant set if the undisturbed system has a minimum convergence rate β. This set is centred at x_* and it has a ball shape with radius $\frac{\delta}{1-\beta}$. It is interesting to observe that the size of the set is proportional to the disturbance magnitude and it is also function of the convergence rate β.

In the following it is presented a theorem which shows that the steepest descent algorithm, with the step size (8), has a minimum convergence rate, under some assumptions on the gradient of the cost function.

Theorem 5.2: *Consider the steepest descent algorithm with the step size sequence (8). Consider that Assumption 2.2 is verified and a set $\mathcal{D} \ni \rho_*$. If the following conditions are respected*

(1)

$$a < \|M(\rho)\| < b \; \forall \rho \in \mathcal{D} \quad (12)$$

(2)

$$M(\rho) > c \; \forall \rho \in \mathcal{D} \quad (13)$$

then

$$\|\rho_{i+1} - \rho_*\|^2 < (1 - \frac{a^2c^2}{b^4})\|\rho_i - \rho_*\|^2 \; \forall \rho_i \in \mathcal{D}$$

Proof Using the rule (8) and the conditions of the theorem, the step size sequence is such that

$$\gamma_i > \frac{c}{b^2} \; \forall \rho_i \in \mathcal{D}.$$

Now, we can show that

$$\|\gamma_i \nabla J(\rho_i)\| = \gamma_i \|\nabla J(\rho_i)\| > \frac{c}{b^2}\|\nabla J(\rho_i)\| > \frac{ac}{b^2}\|\rho_i - \rho_*\| \; \forall \rho_i \in \mathcal{D}. \quad (14)$$

It is easy to show that the rule (8) also ensures that the angle between $(\rho_{i+1} - \rho_*)$ and $\nabla J(\rho_i)$

is always larger than $\frac{\pi}{2} rad$, so

$$\|\rho_{i+1} - \rho_*\|^2 < \|\rho_i - \rho_*\|^2 - \|\gamma_i \nabla J(\rho_i)\|^2. \tag{15}$$

So, using (14) and (15) it is possible to establish the minimum convergence rate of the algorithm

$$\|\rho_{i+1} - \rho_*\|^2 < (1 - \frac{a^2 c^2}{b^4})\|\rho_i - \rho_*\|^2 \ \forall \rho_i \in \mathcal{D}.$$

□

This theorem proves, under some weak conditions about the gradient of the cost function, that the steepest descent algorithm using the step size (8) has a minimum convergence rate $(1 - \frac{a^2c^2}{b^4})$. The conditions of the theorem imply that $\|\rho_i - \rho_*\|a < \|\nabla J(\rho_i)\| < b\|\rho_i - \rho_*\|$ and that the angle between the gradient of the cost function and $(\rho_i - \rho_*)$ is smaller than $\pi/2$ rad. The two implications cannot be considered very conservative from the authors' point of view.

If the result of the Lemma 5.1 is used together with the result of the Theorem 5.2, it is possible to show that the steepest descent algorithm is robust to disturbances.

Corollary 5.3: *Consider that the steepest descent algorithm is perturbed*

$$\rho_{i+1} = \rho_i - \gamma_i \nabla J(\rho_i) + g_i \tag{16}$$

where $\|g_i\| < \delta \ \forall i$. *Consider that* $a < \|M(\rho)\| < b, M(\rho) > c \ \forall \rho \in \mathcal{D}$. *Consider that the step size is chosen by (8). Consider also the set*

$$\mathcal{Z}(\rho_d) = \left\{ \rho : \|(\rho - \rho_d)\| < \frac{\delta b^2}{a^2 c^2} \right\}.$$

and that $\mathcal{Z}(\rho_d) \subset \mathcal{D}$. *Then* $\forall \rho_0 \in \mathcal{D}$ *the disturbed steepest descent algorithm (16) converges to the invariant set* $\mathcal{Z}(x_*)$.

This result's relevance is that it shows that the step size policy presented in (Eckhard and Bazanella 2009) not only ensures convergence to the global minimum with high convergence rate, but also these advantageous properties are kept when there are disturbances affecting the algorithm.

The next section presents simulations that compare the convergence between the disturbed algorithm and the undisturbed one.

6. Simulations

Consider the following system

$$y(t) = \frac{0.1}{z - 0.8} u(t). \tag{17}$$

This system is controlled by a PI controller

$$C(q, \rho) = \begin{bmatrix} \rho_1 & \rho_2 \end{bmatrix} \begin{bmatrix} 1 & \frac{z}{z-1} \end{bmatrix}^T. \tag{18}$$

The following reference model is specified:

$$y_d(t) = \frac{0.1}{z - 0.9} r(t). \tag{19}$$

It is known that the above reference model can be achieved by the controller

$$C_d(q) = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix} \begin{bmatrix} 1 & \frac{z}{z-1} \end{bmatrix}^T. \tag{20}$$

We consider that the system is initially in closed-loop with the controller

$$C(q,\rho) = \begin{bmatrix} 0.7 & 0.15 \end{bmatrix} \begin{bmatrix} 1 & \frac{z}{z-1} \end{bmatrix}^T. \tag{21}$$

We want now to improve the closed-loop performance utilizing the algorithm (6) where $\nabla J(\rho)$ is substituted by the estimate based on data of the method Iterative Feedback Tuning (IFT). The following reference signal, which is conceived to guarantee persistence of excitation, is used to obtain the data:

$$r(t) = square\left(\frac{2\pi t}{256}\right),$$

where $square\left(\frac{2\pi t}{T}\right)$ stands for a square wave with period T.

Figure 1 and Figure 2 show the evolution of the controller parameters for 100 iterations using the proposed sequence of step sizes with disturbance ($\delta = 0.01$) and without disturbance ($\delta = 0$). It is possible to see in the figures that the disturbed algorithm converged to a set that includes the global minimum.

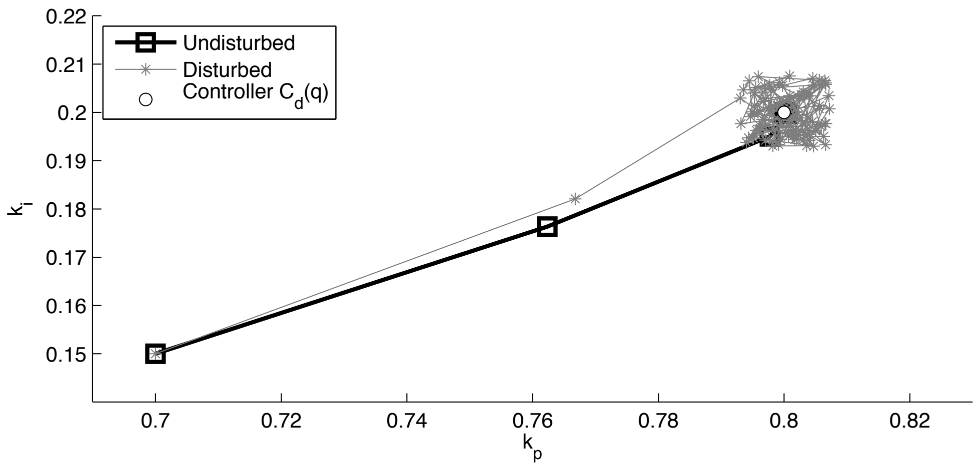

Figure 1. Convergence of the undisturbed and disturbed algorithm.

7. Conclusion

This paper discussed the convergence of the iterative data-based controller tuning when the steepest descent algorithm is disturbed. The classical choice of the step sizes is robust to uncertainties but presents low convergence rates. Some recent results on the choice of the step size present high convergence rate but do not deal with disturbances on the estimates. This work studied these faster algorithms and has shown that when there are disturbances, the algorithm converges to a ball centred at the global minimum. The ball size is function of the perturbation magnitude and of the minimum convergence rate of the undisturbed algorithm. This work dealt only with SISO systems. We propose to extend the results to MIMO systems in a future work.

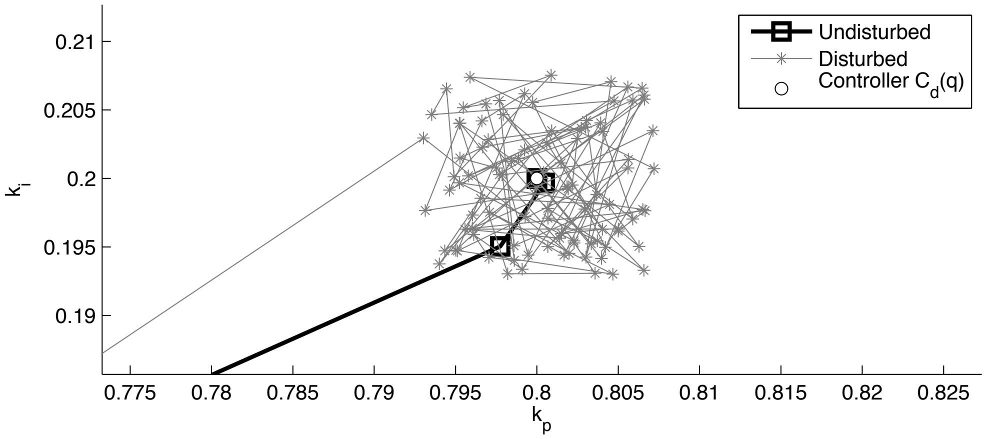

Figure 2. Convergence of the undisturbed and disturbed algorithm (ZOOM).

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