Cost function shaping of the output error criterion

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Abstract

Identification of an output error model using the prediction error method leads to an optimization problem built on input/output data collected from the system to be identified. It is often hard to find the global solution of this optimization problem because in most cases both the corresponding objective function and the search space are nonconvex. The difficulty in solving the optimization problem depends mainly on the experimental conditions, more specifically on the spectra of the input/output data collected from the system. It is therefore possible to improve the convergence of the algorithms by properly choosing the data prefilters; in this paper we show how to perform this choice. We present the application of the proposed approach to case studies where the standard algorithms tend to fail to converge to the global minimum.

Key words: Identification methods, Model fitting.

1 Introduction

The prediction error method (PEM) uses input-output data collected from the process to form a cost function and then estimates the parameters by minimizing this cost function. In this contribution we will consider identification of so called output error (OE) models using PEM; this combination of model structure and identification method is henceforth denoted OE-PEM. Under mild assumptions, the global minimum of the cost function is a consistent estimate of the model parameters and its asymptotic variance attains the Cramér-Rao Bound when the output noise is white. Therefore, identification by means of OE-PEM provides a consistent and otherwise statistically appealing estimate of the system parameters and transfer function, provided that the global minimum of the cost function is found by the optimization procedure [13].

One difficulty in applying OE-PEM is that in many cases achieving the global minimum may prove difficult due to the nonconvexity of the cost function and of the search space [15, 20, 22]. This problem can be mitigated, but not completely solved, by obtaining good initial guesses for the parameters [16, 15]. Another approach, based on input design, has been recently proposed [6, 10]. However, the method has two drawbacks, the design is based on the process model which is unknown and it is necessary to collect data using a specific input signal. In this work we present another strategy, which consists in prefiltering the data so that the cost function presents desirable properties. Filtering reduces the signal energy and hence the information content in the data, so we propose the application of a sequence of filters such that convergence to the global minimum is guaranteed and still all the information contained in the data is extracted. The method presented here is an extension of the conference work [9], which now includes a proof of convergence of the method, extra examples and better explanation of the methodology.

The paper is organized as follows. Section 2 presents basic definitions and the problem formulation. The filtering solution proposed in this paper is based on the properties of the cost function and the same theoretical background from [9], so only the main points of this theory are reviewed in Section 3. The new algorithm, which is based on shaping the cost function by sequentially prefiltering the data, is presented in Section 4. A case study is given in Section 5, and some concluding remarks appear in Section 6.

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2 Preliminaries

Consider the identification of a linear time-invariant discrete-time single-input single-output true system

$$\begin{aligned} y(t) &= G_0(q)u(t) + v(t), \\ v(t) &= H_0(q)e(t). \end{aligned} \tag{1}$$

In (1), q is the forward-shift operator, $y(t)$ is the output signal and $u(t)$ is the input signal.

We will consider the following output-error model structure:

$$y(t,\theta) = G(q,\theta)u(t) + \nu(t) \tag{2}$$

where $\nu(t)$ is assumed to be white noise and

$$G(q,\theta) = \frac{B^T(q)\theta}{1 + F^T(q)\theta} \tag{3}$$

and $B(q), F(q)$ are vectors of given rational transfer functions. The parameter vector θ is assumed to belong to a set Θ, i.e. $\theta \in \Theta \subseteq \mathbb{R}^n$. For every given $\theta \in \Theta$, $G(q,\theta)$ is called a *model*, and the collection of all models, i.e.

$$\mathcal{G} = \{G(\theta) : \ \theta \in \Theta\}, \tag{4}$$

is called the *model set* $\mathcal{G}$.

The assumptions on the true system and the model are given next.

Assumption 1 *In (1), $G_0(q)$ and $H_0(q)$ are stable proper transfer functions with H_0 monic, i.e. $H_0(\infty) = 1$, $e(t)$ is a zero mean white noise sequence with variance σ_e^2 and bounded moments up to order $4+\delta$ for some $\delta > 0$. The input signal is assumed to be a quasi-stationary signal [13], uncorrelated to the noise signal $e(t)$, and whose spectrum, $\Phi_u(\omega)$, is strictly positive in $[-\pi,\pi]$, implying that it is persistently exciting of any order [13].*

The model set $\mathcal{G}$ is a uniformly stable family of stable rational transfer functions [13], with $G(q,\ \cdot\ ) \in C^\infty(\Theta)$, where $\Theta \subseteq \mathbb{R}^n$ is compact. We will also assume that there is a parameter vector $\theta_0 \in \Theta$ such that the corresponding model can describe precisely the process transfer function $G_0(q)$, i.e.

$$G(q,\theta_0) = G_0(q).$$

Furthermore, the model structure $\mathcal{G}$ is globally identifiable at $\theta_0 \in \Theta$.

In the approach we propose, the input and output signals are filtered by a Bounded-Input-Bounded-Output (BIBO-stable) linear low pass filter $L(q)$ before data is used to identify the model parameters. Therefore, introduce the filtered output and input signals

$$y_L(t) = L(q)y(t), \quad u_L(t) = L(q)u(t). \tag{5}$$

Identification of an output error model using PEM based on N input-output data consists in finding, among all the models in the pre-specified model set, the one that solves the following optimization problem:

$$\hat{\theta}_N = \arg\min_{\theta\in\Theta} V_N(\theta) \tag{6}$$

$$V_N(\theta) = \frac{1}{N}\sum_{t=1}^{N}[\hat{y}_L(t,\theta) - y_L(t)]^2 \tag{7}$$

where $\hat{y}_L(t,\theta)$ is the (filtered) optimal one-step-ahead predictor [13]

$$\hat{y}_L(t,\theta) = G(q,\theta)u_L(t). \tag{8}$$

In this paper, we study the asymptotic properties as $N \to \infty$ of the optimization problem defined in (6), through the properties of the limit function (pointwise in $\theta \in \Theta$ with probability 1 [13])

$$\begin{aligned} V(\theta) &= \lim_{N\to\infty} V_N(\theta) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_u(\omega)\left|L(e^{j\omega})(G_0(e^{j\omega}) - G(e^{j\omega},\theta))\right|^2 d\omega \\ &\quad + \frac{1}{2\pi}\int_{-\pi}^{\pi} \sigma_e^2 \left|L(e^{j\omega})H_0(e^{j\omega})\right|^2 d\omega. \end{aligned} \tag{9}$$

where we have applied Parseval's theorem. We also define the asymptotic value of the estimate as $\theta^* = \lim_{N\to\infty}\hat{\theta}_N$. Under Assumption 1, it follows that $\theta^* = \theta_0$. Notice that this and (9) holds even if $v(t)$ is not white as long as u and e are uncorrelated.

3 Properties of the cost function

The prediction error method is based on finding the global minimum of a cost function, so it is important to have a formal characterization of its minima.

Definition 1 *The limit solution θ_0 of the optimization problem (6) as $N \to \infty$ is also called a* ***global minimum*** *of the function $V(\theta)$. A point θ^+ is called a (non-global)* ***local minimum*** *if it is not a global minimum and there exists a $\delta > 0$ such that $V(\theta) \geq V(\theta^+)$ for all θ satisfying $\|\theta - \theta^+\| < \delta$.*

When a cost function is nonconvex, it can be very challenging to create an algorithm to solve problem (6) and find the global minimum θ^*. Most algorithms are based on the gradient of the cost function and it is not uncommon that they converge to local minima or diverge, stopping at the boundary of the search space Θ [18, 2, 19, 9, 7, 8]. For example, please see Figure 1 in Example 1 in Section 5. This behaviour becomes increasingly common as the model order becomes larger.

In this paper we focus in the steepest descent optimization algorithm defined by

$$\theta_{i+1} = \theta_i - \gamma_i \nabla V(\theta_i), \quad i = 1, \ldots \tag{10}$$

where $\gamma_i > 0$ is the step size of the algorithm and $\nabla V(\theta_i)$ is the gradient of the cost function with respect to θ_i (the estimate at iteration i). The algorithm is initialized with some θ_1. The convergence of the algorithm to the global minimum θ_0 depends on its initialization, and a set of initial conditions for which an algorithm converges to a minimum is called a domain of attraction (DOA) of the minimum.

Definition 2 *Let θ_0 be the global minimum of the function $V(\theta)$. A set $\Omega \subset \mathbb{R}^n$ is a* ***domain of attraction*** *of the minimum for the function $V(\theta)$ and a given algorithm if $\lim_{i\to\infty} \theta_i = \theta_0$ for all $\theta_1 \in \Omega$.*

The convergence of the algorithm to the global minimum θ_0 depends on the initial condition θ_1 and the shape of the cost function $V(\theta)$. In this paper we explore how the shape of the cost function affects the DOA, similarly to what has been proposed for controller tuning in [4, 3]. The fundamental concept behind our approach is the *Candidate Domain of Attraction* (CDOA), also known as *Decreasing Euclidean Parameter Error Norm* (DEPEN) region [11, 20, 21], which is defined next.

Definition 3 *Let θ_0 be the global minimum of the function $V(\theta)$. A set Λ is a* ***candidate domain of attraction*** *for the function $V(\theta)$ if $\theta_0 \in \Lambda$ and*

$$(\theta - \theta_0)^T \nabla V(\theta) > 0 \text{ for all } \theta \in \Lambda \text{ such that } \theta \neq \theta_0. \tag{11}$$

For all points in a CDOA the angle α between $\nabla V(\theta)$ and the vector $(\theta - \theta_0)$ is smaller than $\frac{\pi}{2}$ [rad]. This property makes the convergence to the global minimum easier to achieve in gradient-based methods, because the negative gradient is always pointing towards the global minimum, and never away from it. Specifically, the steepest descent algorithm converges to the global minimum if it is initialized inside a ball which is a CDOA, as stated in the next lemma.

Lemma 1 *[4, 6]. Let θ_0 be the global minimum of $V(\theta)$ and define a set $\mathcal{B}(\theta_0) = \left\{\theta : (\theta - \theta_0)^T(\theta - \theta_0) < \alpha\right\}$. If $\mathcal{B}(\theta_0)$ is a CDOA then there exists a sequence $\gamma_i, i = 1, 2, \ldots$ such that $\mathcal{B}(\theta_0)$ is a DOA of algorithm (10) for $J(\theta)$.*

The above lemma shows that if condition (11) is satisfied there is a step size sequence γ_i which ensures the convergence of the algorithm to the global minimum θ_0. Actual convergence also involves the proper choice of the sequence γ_i, an issue which we do not address in this paper. For more information please see [8].

The CDOA is a property of the cost function which is strongly related to the shape of the cost function. The larger is the CDOA, the easier is for the algorithms to converge to the global minimum of the criterion. Using the same approach as in [10], it is possible to obtain an integral form for the condition in (11):

$$(\theta - \theta_0)^T \nabla V(\theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} \Phi_u |L|^2 |G_0 - G(\theta)|^2 \Re \{K(\theta_0, \theta)\} \, d\omega > 0 \tag{12}$$

where we have dropped the frequency dependence to shorten de expression, where $\Re\{\cdot\}$ represents the real part of a complex number and where we have defined $K(e^{j\omega}, \theta_0, \theta)$ as

$$K(e^{j\omega}, \theta_0, \theta) \triangleq \frac{1 + F^T(e^{j\omega})\theta_0}{1 + F^T(e^{j\omega})\theta}. \tag{13}$$

In order for a set Λ to be a CDOA the integral (12) must be positive for all θ inside the set. Observe that the user can manipulate two factors of the integrand: the input spectrum $\Phi_u(\omega)$ and the filter $L(e^{j\omega})$. Choosing carefully these two factors it is possible to ensure the integral is positive and then to enlarge the CDOA set; this is exactly what we propose in this article.

3.1 Input design

The article [10] proposes a method to design the input spectrum $\Phi_u(\omega)$ with objective to improve the convergence of the optimisation algorithms. The method parametrises the input spectrum and solves an optimisation problem where the energy of the input signal is minimised while imposing a bound on the variance of the estimated model parameters. The constraint (12) is also imposed on the optimisation problem, and therefore, the user can choose a set Λ which is a CDOA.

The implementation of these methods requires that a first experiment is run to estimate a rough model, and then the optimal input signal is computed, to be used in a second experiment. The user must run a specific experiment, which is not always possible. It is also usual that the user only has access to data previously collected from the process, and cannot run the optimal experiment.

When the user has the freedom to run many experiments on the process, the input design proposed in [10] is recommend since it improves the convergence of the algorithms and it also reduces the variance of the estimate using the optimal input signal. However, if the data is already collected or if the user cannot choose the input signal, then all he/she can manipulate is the filter $L(q)$ to ensure condition (12) is verified. In this article we propose a method to choose the filter $L(q)$.

3.2 Iterative filtering

Condition (12) is an integral whose integrand is composed by several factors, all of which are non-negative by construction, except for the factor $\Re\left\{K(e^{j\omega},\theta_0,\theta)\right\}$. Thus, whether or not the integral is positive depends exclusively on the function $K(e^{j\omega},\theta_0,\theta)$. The article [9] shows that if all poles and zeros of the transfer function $K(e^{j\omega},\theta_0,\theta)$ are inside the unit circle, both $K(e^{j0},\theta_0,\theta)$ and $K(e^{j\pi},\theta_0,\theta)$ are real and positive. Using this fact and the continuity of the function $K(e^{j\omega},\theta_0,\theta)$ with respect to both ω and θ, as well as the compactness of Θ, results in the following lemma.

Lemma 2 *There exist constants $\omega_l,\omega_h \in (0,\pi)$ and $\delta > 0$ such that for all $\theta \in \Theta$:*

$$\Re\left\{K(e^{j\omega},\theta_0,\theta)\right\} > \delta, \quad 0 \le \omega \le \omega_l,$$
$$\Re\left\{K(e^{j\omega},\theta_0,\theta)\right\} > \delta, \quad \omega_h \le \omega \le \pi. \tag{14}$$

The integrand of (12) is thus always positive for sufficiently low or high frequencies. Hence, if this integrand has large magnitude at these ranges the whole integral is positive, as desired. This property, in turn, can be enforced by properly choosing the filter $L(q)$, so as to emphasize low and/or high frequencies.

Therefore, using the filter $L(q)$ one can ensure that the set Θ is a CDOA which makes it easier for gradient-based algorithms to converge to the global minimum θ_0. The result is simple and powerful but of course it does not come for free. By filtering out some frequencies from the data, the total energy of the signals will be reduced, thus increasing the variance of the resulting estimate, an increase that can be dramatic. Remember that all the results came from the analysis of the asymptotic cost function $V(\theta)$. When a finite data set is utilised then the noise on the output will affect the estimate even more if the input energy is reduced. To overcome this problem, we propose the application of a bank of successive filters, as described in the next section.

4 The algorithm

To overcome the difficulties of optimizing the OE-PEM cost function, the following homotopy method is proposed:

(1) Choose $M \in \mathbb{N}$ (number of pre-filters) and $\hat{\theta}_N^0 \in \Theta$ (initial condition).
(2) For $i = 1,\ldots,M$, do
  (a) Pre-filter the data Z^N by a low pass filter [1] $L^{\omega_i}(q)$ of bandwidth $\omega_i = \pi i/M$.
  (b) Compute the OE-PEM estimate $\hat{\theta}_N^i$ based on the pre-filtered data, using a gradient scheme starting from the initial condition $\hat{\theta}_N^{i-1}$.
(3) Define the final estimate as $\hat{\theta}_N \triangleq \hat{\theta}_N^M$.

The family of low pass filters to be considered in this algorithm is assumed to satisfy the following conditions:

Assumption 2 *$\{L^\omega(q)\}$ and $\{\partial L^\omega(q)/\partial\omega\}$ are uniformly stable families of rational transfer functions whose coefficients are continuously twice differentiable in $\omega \in [0,\pi]$. Furthermore, $L^\omega(q) \to 1$ weakly as $\omega \to 1$ (*i.e., *$\int_{-\pi}^{\pi}|L^\omega(e^{j\tau})|^2 f(\tau)d\tau \to \int_{-\pi}^{\pi} f(\tau)d\tau$ as $\omega \to 1$ for all continuous functions $f: [-\pi,\pi] \to \mathbb{R}_0^+$), and for all $\delta \in (0,\pi)$ it holds that $\int_\delta^\pi |L^\omega(e^{j\tau})|^2 d\tau \Big/ \int_0^\delta |L^\omega(e^{j\tau})|^2 d\tau \to 0$ as $\omega \to 0$.*

The homotopy method possesses the following asymptotic convergence property:

Theorem 3 *Under Assumption 1 and 2, there is an $M_0 \in \mathbb{N}$ such that the estimate $\hat{\theta}_N$ provided by the homotopy method converges to θ_0 as $N \to \infty$ almost surely for all $M \ge M_0$, irrespective of $\hat{\theta}_N^0$.*

PROOF. Let us denote by V_N^ω the OE-PEM cost function when the data Z^N has been pre-filtered by a low pass filter L^ω of bandwidth ω, and let us define $\bar{V}^\omega(\theta)$ as the pointwise limit (in θ) $\lim_{N\to\infty} V_N^\omega(\theta)$. Notice that V_N^π corresponds to the OE-PEM cost function applied to the unfiltered data. In order to establish this result, we split the proof into several steps:

(1) Show that there is an $\varepsilon > 0$ such that the region $\{\theta \in \Theta : \|\theta-\theta_0\|_2 < \varepsilon\}$ is a CDOA for the function $\bar{V}^\omega$ for all $\omega \in (0,\pi]$.
(2) Establish that, given any $\underline{\omega} \in (0,\pi)$, with probability one, there exists an $N_0 \in \mathbb{N}$ (dependent on the realization of Z^N and on θ_0) such that for all $N \ge N_0$, the set $\{\theta \in \Theta : \|\theta-\theta_0\|_2 < \varepsilon/2\}$ is a CDOA for the function V_N^ω for all $\omega \in (\underline{\omega},\pi]$.
(3) Show that there exists an $M_0 \in \mathbb{N}$ such that for all $M \ge M_0$, with probability one, there is an $N_1 \in \mathbb{N}$ such that Θ is a CDOA for the function $V_N^{\pi/M}$ for all $N \ge N_1$.
(4) Conclude that, with probability one, the estimate $\hat{\theta}_N$ provided by the homotopy method for the given value of M satisfies $\|\hat{\theta}_N-\theta_0\|_2 < \varepsilon/2$.

As ε can be made arbitrarily small, the preceding steps would establish the convergence of the homotopy-based estimator. In the sequel, the aforementioned steps are detailed:

Step 1. It is known that, under the stated assumptions, θ_0 is the unique global minimum of $\bar{V}^\omega$ for all $\omega \in (0,\pi]$;

[1] In this section we have added the superscript ω to the prefilter L to emphasize the dependence on the bandwidth ω.

this is due to the fact that G_0 and G are analytic in a neighbourhood of the unit circle, hence if $L^\omega(G_0 - G) \equiv 0$, which means that $G_0(e^{j\omega}) - G(e^{j\omega}, \theta) = 0$ in a sub-interval of $[-\pi, \pi]$, then $G_0 - G \equiv 0$. Furthermore, according to Lemma 2 and Assumption 2 there is a $\bar{\omega} \in (0, \pi]$ such that θ_0 is the only local minimizer of $\bar{V}^\omega$ for all $\omega \in (0, \bar{\omega}]$; this follows from the fact that by making the bandwidth of the prefilter, ω, sufficiently small, the integral in (12) is strictly greater than 0 for every fixed $\theta \in \Theta \setminus \{\theta_0\}$. Due to the persistence of excitation of the data and the global identifiability of $\mathcal{G}$, it follows that

$$\frac{\partial^2 \bar{V}^\omega}{\partial\theta\partial\theta^T}(\theta_0) > 0, \quad \forall \omega \in [\bar{\omega}, \pi], \tag{15}$$

i.e., the smallest eigenvalue of the Hessian of $\bar{V}^\omega$ at θ_0 is strictly positive for all $\omega \in [\bar{\omega}, \pi]$. In addition, due to the rational structure of L^ω and the continuously differentiable dependence of the coefficients of L^ω on the bandwidth ω, such eigenvalue is a continuous function of ω in that range. Now, let us denote by $\delta(\omega)$ the infimum of the Euclidean distance between θ_0 and any other critical point of $\bar{V}^\omega$ in Θ, for $\omega \in [\bar{\omega}, \pi]$. Evidently, $\delta(\omega) > 0$. If $\inf_{\omega \in [\bar{\omega}, \pi]} \delta(\omega) = 0$, then $\lambda_{\min}(\partial^2 \bar{V}^\omega(\theta_0)/\partial\theta\partial\theta^T) = 0$ for some $\omega \in [\bar{\omega}, \pi]$, which contradicts (15). This means that $\inf_{\omega \in [\bar{\omega}, \pi]} \delta(\omega) > 0$, and we can take

$$\varepsilon = \frac{1}{2} \inf_{\omega \in [\bar{\omega}, \pi]} \delta(\omega).$$

Step 2. Under the stated assumptions, Lemma 3.1 of [12] can be applied to conclude the uniform convergence of $V_N^\omega(\theta)$ (and its first two derivatives with respect to θ) over $\theta \in \Theta$ and $\omega \in [\underline{\omega}, \pi]$, *i.e.*, where ω is included as an additional "parameter". This means that

$$\begin{aligned}
&\lim_{N\to\infty} \sup_{\substack{\omega \in [\underline{\omega}, \pi] \\ \theta \in \Theta}} \left| V_N^\omega(\theta) - \bar{V}^\omega(\theta) \right| = 0 \quad \text{w.p.1} \\
&\lim_{N\to\infty} \sup_{\substack{\omega \in [\underline{\omega}, \pi] \\ \theta \in \Theta}} \left\| \frac{\partial V_N^\omega(\theta)}{\partial\theta} - \frac{\partial \bar{V}^\omega(\theta)}{\partial\theta} \right\|_2 = 0 \quad \text{w.p.1} \\
&\lim_{N\to\infty} \sup_{\substack{\omega \in [\underline{\omega}, \pi] \\ \theta \in \Theta}} \left\| \frac{\partial^2 V_N^\omega(\theta)}{\partial\theta\partial\theta^T} - \frac{\partial^2 \bar{V}^\omega(\theta)}{\partial\theta\partial\theta^T} \right\|_2 = 0 \quad \text{w.p.1.}
\end{aligned} \tag{16}$$

These results imply the statement of Step 2.

Step 3. If we take any $\underline{\omega} \in (0, \omega_l/2)$ (where ω_l is defined in Lemma 2), then equations (16) imply that with probability one, there is an $N_1 \in \mathbb{N}$ such that for all $N \geq N_1$, Θ is a CDOA for the function V_N^ω for all $\omega \in (\underline{\omega}, \omega_l/2)$ [2]. Therefore, the statement of Step 3 holds for every $M \in \mathbb{N}$ such that $\pi/M < \omega_l/2$.

[2] The need for defining $\underline{\omega} > 0$ come from the fact that V_N^0 is constant, i.e., it does not have a unique global minimum.

Step 4. By taking M as in Step 3, and any $N \geq \max\{N_0, N_1\}$, it follows from Steps 2 and 3 that $\|\hat{\theta}_N^1 - \theta_0\|_2 < \varepsilon/2$. Proceeding by induction on $i \in \{1, \dots, M-1\}$, it can be seen from Step 2 that if $\|\hat{\theta}_N^i - \theta_0\|_2 < \varepsilon/2$ then $\|\hat{\theta}_N^{i+1} - \theta_0\|_2 < \varepsilon/2$. This establishes Step 4, and concludes the proof. □

The dependence of the value of M_0 on the (unknown) true parameter vector θ_0 can be avoided by a standard compactness argument, as shown in the following corollary.

Corollary 4 *There exists an $M_0 \in \mathbb{N}$ independent of the value of θ_0 for which the statement of Theorem 3 holds.*

PROOF. Define, for each $m \in \mathbb{N}$, the set

$$\Theta_m := \{\theta \in \Theta : \text{ Theorem 3 holds for } \theta_0 = \theta \text{ with } M_0 = m\}.$$

It can be seen that $\{\Theta_m : m \in \mathbb{N}\}$ is an open cover of Θ. The compactness of Θ then implies the existence of a finite subcover $\{\Theta_{m_i} : 1 \leq i \leq n\}$ of Θ, where $n \in \mathbb{N}$. The corollary follows by taking $M_0 = \max\{m_i : 1 \leq i \leq n\}$. □

Remark 1 *Corollary 4 establishes the existence of M_0, but it does not provide a value for it. The smallest value of M_0 can in principle be computed from the following semi-infinite optimization problem:*

$$\begin{aligned}
\min_{M_0 \in \mathbb{N}} \quad & M_0 \\
\text{s.t.} \quad & \int_{-\pi}^{\pi} \Phi_u \left| L^{\pi/M_0} \right|^2 |G(\theta_0) - G(\theta)|^2 \cdot \\
& \cdot \Re\left\{ K(e^{j\tau}, \theta_0, \theta) \right\} d\tau > 0, \quad \textit{for all } \theta, \theta_0 \in \Theta, \theta \neq \theta_0.
\end{aligned}$$

This optimization problem can be approximated, by discretizing the constraint, prior to performing an experiment, since it does not depend on experimental data. We do not pursue this approach further in this paper, however, due to reasons of space.

5 Case Studies

5.1 Example 1 - One dimensional

Consider the example where θ is a scalar, $L = 1$, $\sigma_e = 0$, $u(t) = 70\sin(t/20) + 140\sin(t/5)$,

$$\begin{aligned}
G_0(q) &= \frac{0.006q^{-3}}{(1 - 0.9q^{-1})(1 - 0.8q^{-1})(1 - 0.7q^{-1})} \\
&= \frac{0.006q^{-3}}{1 - 2.4q^{-1} + 1.91q^{-2} - 0.504q^{-3}}
\end{aligned}$$

and

$$G(\theta) = (0.006q^{-3})/((1-2.4q^{-1}+1.91q^{-2}-0.504q^{-3}) + \theta(-0.16q^{-1}+0.26q^{-2}+0.46q^{-3})).$$

The model $G(\theta)$ is stable for $\theta \in \Theta = [-0.06 \quad 0.032]$ and the minimum of the function occurs when $\theta = 0$. The cost function $V(\theta)$ is shown in Figure 1 in blue. Observe that the function has only one minimum, but any gradient based algorithm will converge to the boundary of the search space (0.032) when initialized with $\theta_1 > 0.0229$. Observe that this cost function does not possess local minima, but still it is nonconvex and application of standard optimization to it may fail.

Consider now that the filter $L(q) = \frac{0.0155+0.0155q^{-1}}{1-0.9691q^{-1}}$ (first order Butterworth filter) is used to modify the input/output data and shape the cost function. The resulting cost function is also shown in Figure 1 in red. Observe that now the steepest descent algorithm can converge to the global minimum for any initial condition inside the set Θ.

The filter has changed the cost function and enlarged the candidate domain of attraction. However, as it is possible to see in the figure, the cost function becomes flatter, and this indicates that the variance of the estimate will be larger, when the filtered data is used.

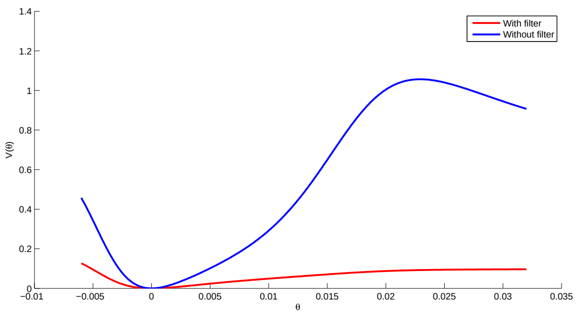

Fig. 1. Cost function $V(\theta)$ with and without filter.

5.2 *Example 2 - Simple example*

Consider the case where the process is described by

$$G_0(q) = \frac{q^{-1} - 0.9q^{-2}}{(1-0.85q^{-1})(1-0.95q^{-1})} = \frac{q^{-1} - 0.9q^{-2}}{1 - 1.8q^{-1} + 0.8075q^{-2}},$$

$$H_0(q) = \frac{1-0.8q^{-1}}{1-0.7q^{-1}}, \tag{17}$$

with noise variance $\sigma_e = 1$ and that $u(t)$ is Gaussian noise with unitary variance. The length of the input/output data set is $N = 1000$ samples.

The model is described by

$$G(\theta) = \frac{\theta_1 q^{-1} + \theta_2}{1 + \theta_3 q^{-1} + \theta_4 q^{-2}} = \frac{B^T(q)\theta}{1 + F^T(q)\theta}$$

where

$$B(q) = [q^{-1} \; q^{-2} \;\; 0 \; 0\,]^T, \quad F(q) = [0 \; 0 \; q^{-1} \; q^{-2}]^T.$$

Observe that the model $G(q,\theta)$ can exactly describe the real system $G_0(q)$ when $\theta = \theta_0 = [1 \; -0.9 \; -1.8 \; 0.8075]^T$.

The parameters of the model are identified using the steepest descent algorithm where the initial condition is obtained using the standard least-squares algorithm (which provides biased estimates for the parameters). The procedure is repeated 100 times, so we have obtained 100 different estimates to the model. A histogram of the parameters is plotted in Figure 2, where it is possible to observe that the estimates are very disperse. Only 55 estimates converged to points close to θ_0 while the others 45 converged to points distant from the global minimum.

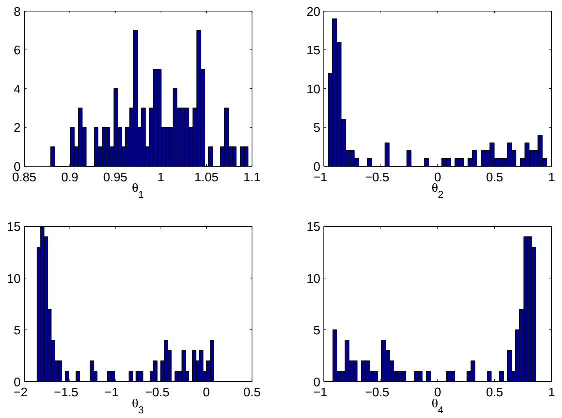

Fig. 2. Histogram of parameter estimates in Example 2 when no prefiltering is used.

Let's now use the proposed iterative algorithm, where the data is initially filtered by a first order Butterworth filter $L(q) = \frac{0.1356+0.1367q^{-1}}{1-0.7265q^{-1}}$. The steepest descent is used again, and the obtained estimates (with filter) are used as initial condition again on the steepest descent algorithm, but now with the original data (without filter). The procedure was again repeated 100 times and the estimates formed the histogram in Figure 3. Now, all the estimates converged to points close to the global minimum θ_0.

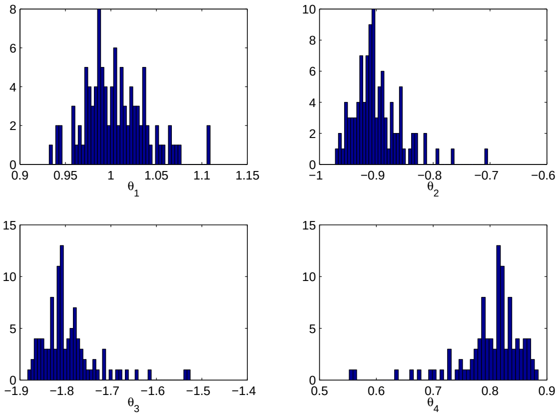

Fig. 3. Histogram of parameter estimates with proposed algorithm.

5.3 Example 3 - A somewhat more complex example

Consider the true system described by

$$G(q,\theta_0) = \frac{B^T(q)\theta_0}{1+F^T(q)\theta_0}, \quad H_0(q) = 1, \quad \sigma_e^2 = 0.01,$$

where

$$B(q) = [q^{-1}\ q^{-2}\ q^{-3}\ q^{-4}\ q^{-5}\ q^{-6}\ q^{-7}\ q^{-8}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ ]^T,$$
$$F(q) = [0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ q^{-1}\ q^{-2}\ q^{-3}\ q^{-4}\ q^{-5}\ q^{-6}\ q^{-7}\ q^{-8}]^T.$$
$$\theta_0 = [-0.2073\ 0.1815\ 1.352\ -3.356\ 3.061\ -1.045\ -0.07957\ 0.0944\ -6.549\ 19.2\ -32.83\ 35.77\ -25.39\ 11.45\ -3\ 0.3491]^T.$$

This model represents a power system consisting of two synchronous generators connected to the grid, with data taken from [1]. The transfer function $G_0(q)$ describes the relation between the field voltage applied to the first generator and the angular speed of the same generator at a given operating condition.

Let us identify a model of this system with an output error model structure

$$G(q,\theta) = \frac{B^T(q)\theta}{1+F^T(q)\theta}$$

The model set $\mathcal{G}$ is formed by all stable models of the form shown above and therefore the parameter vector θ should be constrained to the set Θ such that $G(q,\theta)$ is stable. Observe that the true system belongs to this model set. The iterative filtering approach proposed in this paper has been applied to this problem. At each iteration the input/output data were filtered by a 20th order low-pass FIR filter designed by a window-method [3]. Six iterations of the proposed algorithm were performed, where the cut-off frequencies of the filters were 0.4π, 0.5π, 0.6π, 0.7π, 0.8π and 0.9π (rad/s) respectively.

For comparison, the parameter θ has also been estimated using the Matlab toolboxes **unit** [17] and **ident** [14], besides the algorithm proposed in Section 4. The **unit** toolbox was used with the following commands

```
m.delay=1; m.nA=8; m.nB=7; model=est(z,m);
```

and the **ident** toolbox was used with the following command: `model = oe(data, [8 8 1]);`. The proposed method used the **ident** toolbox, where the data was filtered as specified above.

In the three cases the same input signal $u(t)$ has been applied to generate the data: a white noise sequence with unit variance ($\sigma_u^2 = 1$) and $N = 1000$ samples.

For each one of the three algorithms, 100 Monte-Carlo runs have been performed, thus providing 100 parameter estimates with each method. The results obtained with the toolbox **unit** are summarized in Figure 4, where the vertical axis presents the value of each one of the sixteen elements of the parameter vector θ and the horizontal axis corresponds to the Monte Carlo runs. The Monte-Carlo runs were re-ordered in terms of the smallest θ. It is seen that the first 57 points of the graph present values close to the true parameter value θ_0, indicating that in these 57 runs the algorithm converged to the global minimum of the cost function. But in the remaining 43 Monte Carlo runs the values are close to another point in the parameter space, far away from the true value θ_0, indicating that the algorithm has not yielded the global minimum of the cost function. The **boxplot** of the parameters is shown in Figure 5. On each box, the central mark is the median, the edges of the box are the 0.25 and 0.75 percentiles, the whiskers extend to the most extreme data points not considered outliers, and outliers are plotted individually.

In Section 3 we have stated that when the model has an OE structure and the input is white noise then the cost function has only one minimum (the global one). Still, due to the nonconvexity of the cost function and search space, the **unit** algorithm had trouble in finding the global minimum, even though there is no other minimum. In this example the **unit** algorithm could not find the global minimum in 43 runs.

Identification of this system using the Matlab toolbox **ident** yields results quite similar to those obtained with the toolbox **unit**, providing convergence to the global minimum of the cost function in 26 out of 100 Monte Carlo runs.

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[3] The command `fir1` of Matlab was used.

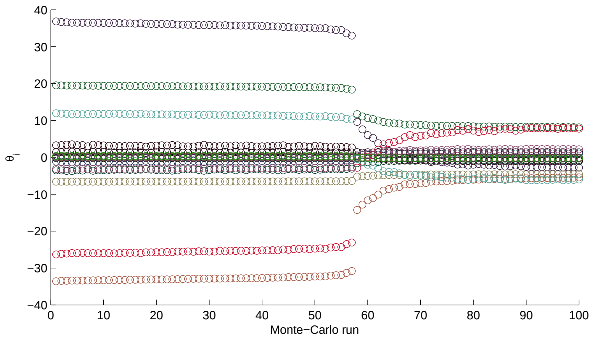

Fig. 4. Monte-Carlo runs with the toolbox `unit`. The plot shows the value of the estimated parameter vector at each run. Each color represents one element of the parameter vector.

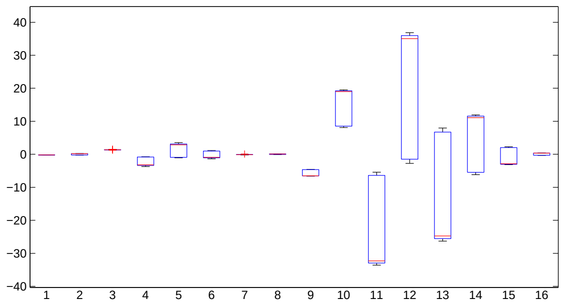

Fig. 5. Boxplot of Monte-Carlo runs with the toolbox `unit`. Each box represents one element of the parameter vector. On each box, the central mark is the median, the edges of the box are the 25th and 75th percentiles, the whiskers extend to the most extreme data points not considered outliers, and outliers are plotted individually.

When the proposed algorithm was used, the convergence to the global minimum has been achieved for every run. Figure 6 shows the boxplots of the parameters for the proposed algorithm.

In order to further illustrate the results, we also compute, for each method, the mean value $\theta_m = \frac{1}{100}\sum_{i=1}^{100}\hat{\theta}_i$ of the model's parameters, where each $\hat{\theta}_i$ represents the estimate obtained at the i-th Monte Carlo run. The mean value of the model parameters obtained with the toolbox `unit`, the toolbox `ident` and the proposed method are,

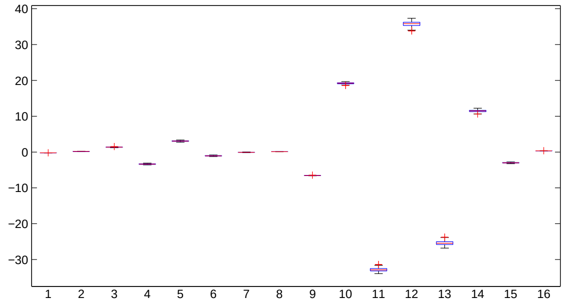

Fig. 6. Boxplot of Monte-Carlo runs with the proposed algorithm.

respectively:

$$\theta_m^{unit} = [-0.2071\ -0.0360\ 1.3298\ -1.9507\ 0.9517\ -0.0396\ -0.0378\ -0.0276\ -5.5050\ 13.3882\ -18.5887\ 15.9136\ -8.4405\ 2.6183\ -0.4008\ 0.0164]^T$$

$$\theta_m^{ident} = [-0.2065\ -0.2952\ 0.8922\ -0.5885\ 0.3352\ -0.2073\ -0.0484\ 0.0841\ -4.2033\ 7.9007\ -8.7065\ 6.4202\ -3.5476\ 1.5685\ -0.5030\ 0.0858]^T$$

$$\theta_m^{prop} = [-0.2073\ 0.1794\ 1.3472\ -3.3269\ 3.0156\ -1.0138\ -0.0910\ 0.0974\ -6.5355\ 19.1230\ -32.6493\ 35.5180\ -25.1770\ 11.3441\ -2.9679\ 0.3450]^T$$

Comparing these values with the true parameter θ_0 we see that the estimates obtained with the proposed method are, on average, much closer to the true parameter values.

5.4 Example 4 - A really more complex example

The proposed method was also tested in a more complex case that resembles the practical conditions where one or more assumptions assumed in this work are not satisfied. The data set used came from [5], which consists of sets of input/output data from $30th$ order systems with small data length N and large noise variance. We have used four collections of data sets:

- S1D1: 100 sets of input/output data from fast systems with $SNR = 10$ and $N = 500$
- S2D1: 100 sets of input/output data from slow systems with $SNR = 10$ and $N = 500$
- S1D2: 100 sets of input/output data fast systems with $SNR = 1$ and $N = 375$
- S2D2: 100 sets of input/output data slow systems with $SNR = 1$ and $N = 375$

The "fast" systems have all their poles inside a circle with radius 0.95 and the "slow" systems have at least one pole outside the circle with radius 0.95 (but inside the unit circle). Each one of the 400 data sets were used to estimated a model for the system and the quality of the model was computed using the FIT measure:

$$FIT = \left(1 - \sqrt{\frac{\sum_{k=1}^{125} \left(g_k^0 - \hat{g}_k\right)^2}{\sum_{k=1}^{125} \left(g_k^0 - \bar{g}^0\right)^2}}\right), \quad \bar{g}^0 = \frac{1}{125}\sum_{k=1}^{125} g_k^0,$$

where g_k^0 is the impulse response of the system and $\hat{g}_k$ is the impulse response of the model. The FIT evaluates 1 only if the impulse response of the model is identical to the impulse response of the system for the first 125 samples, otherwise it evaluates a lower value. The FIT measure was used in [5] to compare the estimate of the models using different identification techniques, and different model orders.

In this article, we estimated models of order 5, 15 and 30 using the `ident` toolbox and the algorithm proposed in this article. The `ident` model was obtained with command `model = oe(data, [n n 1]);` and the proposed method was used with standard steepest descent algorithm on the PEM cost function. The algorithm used adaptive step size to ensure that the cost is reduced at each iteration and used $1,000,000$ steps. The initial condition of the proposed method was obtained using standard least-squares method which gives biased estimates since the noise is white. We have opted to use only one low-pass filter $L(q) = \frac{0.0155+0.0155q^{-1}}{1-0.9691q^{-1}}$ (first order Butterworth filter), to shape the cost function and enlarge the candidate domain of attraction.

The FIT was computed for each model and the mean was computed and presented in Table 1. When the model order is 5, the mean values for all data sets (S1D1, S1D2, S2D1 and S2D2) are higher for the proposed algorithm. When the model order is 15, the mean values for the data sets (S1D1, S2D1 and S2D2) are higher for the proposed algorithm. Using the data set S1D2 the proposed algorithm achieved a lower mean value (less than 2% smaller). When the model order is 30 and then the Assumption 1 is satisfied, for all data sets (S1D1, S1D2, S2D1 and S2D2) the mean of the FIT is higher for the proposed algorithm. In this case the mean of the FIT is much higher with the proposed algorithm than with `ident`, which indicates that the `ident` had trouble in finding the global minimum of the criterion with the high order models.

In Figure 7 we can see the distribution of FIT for each data set, with the different algorithms, for 30th order models. We can observe that not only the mean of the FIT is smaller with the proposed algorithm, but also the variance of the measure.

6 Conclusion

This paper discussed the convergence problems associated with the non-convex optimization problem present in identifying an output error model using the prediction error method. Standard gradient-based optimization algorithms have difficulties to handle this issue. This work has shown that the shape of the cost function depends on the spectra of the input/output data collected from an experiment. Therefore, the cost function may be shaped to facilitate the convergence of optimization algorithms to the global minimum using an adequate choice of the input/output spectra. We have shown that if the input/output spectra are concentrated to low and high frequencies then it is easier to converge to the global minimum of the cost function. Then, an iterative filtering was proposed to change the spectra of the input/output data in order to change the shape of the cost function and facilitate the convergence. Conditions for the convergence of the proposed iterative algorithm were then established and a case study showed that this approach can be very effective in guaranteeing convergence of OE-PEM to the global minimum of its optimization criterion, even in cases where standard identification methods may fail.

Table 1
Mean of FIT different model order and techniques

	S1D1	S2D1	S1D2	S2D2
`ident` - order 5	0.8698	0.7256	0.7193	0.5207
proposed - order 5	0.8917	0.7382	0.7524	0.6091
`ident` - order 15	0.8676	0.7311	0.5701	0.4363
proposed - order 15	0.8939	0.8190	0.5608	0.4821
`ident` - order 30	0.6847	0.5361	0.3175	0.0297
proposed - order 30	0.8483	0.8127	0.3474	0.2863

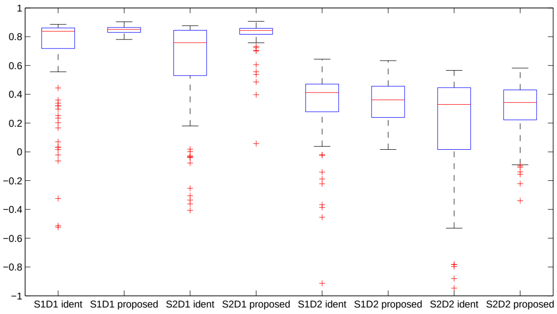

Fig. 7. Boxplot of FIT of models of order 30 with `ident` and the proposed method.

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