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Published version: http://dx.doi.org/10.1049/iet-cta.2008.0221

Finite $\mathcal{L}_2$ gain and internal stabilization of linear systems subject to actuator and sensor saturations*

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Abstract

This paper addresses the control of linear systems subject to both sensor and actuator saturations and additive $\mathcal{L}_2$-bounded disturbances. Supposing that only the output of the linear plant is measurable, the synthesis of stabilizing output feedback dynamic controllers, allowing to ensure the internal closed-loop stability and the finite $\mathcal{L}_2$-gain stabilization, is considered. In this case, it is shown that the closed-loop system presents a nested saturation term. Therefore, based on the use of some modified sector conditions and appropriate variable changes, synthesis conditions in a "quasi"-LMI form are stated in both regional (local) as well as global stability contexts. Different LMI-based optimization problems for computing a controller in order to maximize the disturbance tolerance, the disturbance rejection or the region of stability of the closed-loop system are proposed.

1 Introduction

In practical control systems it is well-known that physical, safety or technological constraints generally induce that the actuators and sensors cannot provide unlimited amplitude signals. In this case, the outputs of both sensors and actuators can saturate. In particular, while the actuators outputs are saturated, the controlled plant operates in open-loop. On the other hand, the saturation of the sensor output induces an incorrect action of the controller, since the actual state or output of the plant is no longer precisely measured. Neglecting actuator and sensor limitations can therefore be source of undesirable or even catastrophic behaviors for the closed-loop system (as the lost of the closed-loop stability). Combat aircrafts and launchers are examples of systems that illustrate the control difficulties due to these major constraints [22], [24].

During the recent past years, the control of linear systems subject to amplitude constrained actuators have been extensively studied in the literature (see for instance [2], [27], [26], [16], [13] and references therein for a general overview). In particular, we can identify works concerning stability analysis and/or

*This work was partially supported by the French-Brazilian research cooperation programs CAPES-STICamSud and CAPES-COFECUB.

†G. Garcia is also with INSA, Toulouse, France.

‡Author partially supported by CNPq, Brazil.

§Author supported by CNPq, Brazil.

stabilization [18], [14], [12], [1], anti-windup synthesis [33], [30] and also the simultaneous synthesis of dynamic output feedback controllers and static anti-windup loops [23], [21], [10], [6].

On the other hand, few results concern the case of sensor saturation. We can cite, for instance, [19], where the effects of sensor saturation on plant observability are studied, [20], in which the global stabilization of a linear SISO system is carried out via the use of dead beat controllers, [3], where the design of an output H_∞ feedback controller for linear systems with sensor nonlinearity is based in the use of a classical sector condition, and also [31], where an anti-windup strategy is proposed considering the guarantee of the global stability of the closed-loop system and the optimization of an $\mathcal{L}_2$ performance criterion. Finally, we can cite [15] in which results addressing the design of a saturated dynamic controller for a system subject to sensor saturation is proposed in a global context of stability.

Since in practice sensors and actuators present amplitude constraints (as it is the case, for instance, in aerospace applications and vibration control), techniques considering the stabilization taking into account simultaneously actuator and sensor saturation are of major interest. However, if few results are available on systems with sensor saturation, still less results concern the case of systems with both actuator and sensor limitations. In [7], adaptive integral control design for linear systems with actuator and sensor nonlinearities is addressed by considering the asymptotic stability of the open-loop. In [9], different methods of analysis and synthesis are presented, in particular relative to PID controllers. On the other hand, the methods proposed in [7] and [9] do not offer a systematic way to deal with the problem.

This paper aims at filling this gap. The paper focus on the problem of controlling linear systems with both sensor and actuator subject to saturation. Moreover, the system is considered to be subject to additive $\mathcal{L}_2$-bounded disturbances. This work can be seen as an extension of the conference paper [8], where preliminary results have been stated for free disturbance systems. Supposing that only the output of the linear plant is measurable, the synthesis of output feedback dynamic controllers satisfying both closed-loop stability and performance requirements is provided. More precisely, the design objective is to ensure the internal closed-loop stability (when the system is disturbance free) and the finite $\mathcal{L}_2$-gain stabilization (in presence of disturbance). It is shown that, in this case, the closed-loop system presents a nested saturation term. Therefore, based on the use of some modified sector conditions and appropriate variable changes, synthesis conditions in a "quasi"-LMI form are stated in both regional (local) as well as global contexts. Depending of the regional or global contexts, different LMI-based optimization problems for computing a controller are proposed in order to maximize the disturbance tolerance, the disturbance rejection or the region of stability of the closed-loop system.

Comparing with previous approaches, it should be emphasized that the proposed approach in the paper complements and improves the techniques developed in [10], [6], [32], where only actuator saturation are considered, and in [3], where only sensor saturation is considered. Moreover, differently from [7] and [9], the approach developed in the paper provides constructive conditions and a systematic procedure to compute the controller.

The paper is organized as follows. The addressed problem is formally stated in section 2. Some preliminaries, needed to the statement of the main results are given in section 3. In particular, some modified sector conditions regarding nested deadzone nonlinearities are presented. Section 4 is dedicated to the main results of the paper, concerning both regional and global stabilization conditions. Computational issues are discussed in section 5. In particular, LMI-based optimization problems are proposed in order to compute the controller from the stated main results. Numerical examples, illustrating the application of the approach, are presented in section 6. The paper ends by a conclusion giving some perspectives.

Notation. For any vector $x \in \Re^n$, $x \succeq 0$ means that all components of x denoted $x_{(i)}$ are nonnegative. For two vectors $x, y \in \Re^n$, the notation $x \succeq y$ means that $x_{(i)} - y_{(i)} \geq 0$, for all $i = 1, ..., n$. The elements of a matrix $A \in \Re^{m \times n}$ are denoted by $A_{(i,j)},\ i = 1, ..., m,\ j = 1, ..., n$. $A_{(i)}$ denotes the ith row of matrix A and $A^{(j)}$ denotes the jth column of matrix A. For two symmetric matrices, A and B, $A > B$ means that $A - B$ is positive definite. A' denotes the transpose of A. $He\{A\} = A' + A$. $Diag(x; y; ...)$ denotes the diagonal matrix obtained from vectors or matrices $x, y, \ldots$. The identity matrix of order n is denoted by I_n. The null matrix of any order is simply denoted by 0. Furthermore, in the case of partitioned symmetric matrices, the symbol $\bullet$ denotes symmetric blocks. For $v \in \Re^m$, $sat_{v_0}(v) : \Re^m \to \Re^m$ denotes the classical symmetric saturation function defined as $(sat_{v_0}(v))_{(i)} = sat_{v_0}(v_{(i)}) = sign(v_{(i)}) \min(v_{0(i)}, |v_{(i)}|)$, $\forall i = 1, ..., m$, where $v_{0(i)} > 0$ denotes the ith amplitude bound.

2 Problem statement

Consider the following continuous-time system:

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) + B_w w(t) \\ y(t) & = & sat_{y_0}(Cx(t)) \\ u(t) & = & sat_{u_0}(y_c(t)) \\ z(t) & = & C_z x(t) + D_z u(t) \end{array} \tag{1}$$

where $x \in \Re^n$ is the state, $u \in \Re^m$ is the plant control input, $y \in \Re^p$ is the measured output and $y_c \in \Re^m$ is the input of the actuator, $z \in \Re^r$ is the regulated output and $w \in \Re^q$ is a disturbance signal. A, B, B_w, C, C_z and D_z are constant matrices of appropriate dimensions. Due to physical limitations both the actuator and sensor outputs are supposed to be bounded in amplitude. Hence, both $u(t)$ and $y(t)$ are saturating signals, with bounds given respectively by the componentwise vectors $u_0 \in \Re^m$ and $y_0 \in \Re^p$. Pairs (A, B) and (C, A) are supposed stabilizable and detectable, respectively. The disturbance vector w is assumed to be limited in energy, that is, $w(t) \in \mathcal{L}_2$ and for some scalar δ, $0 \leq \frac{1}{\delta} < \infty$, it follows that:

$$\|w\|_2^2 = \int_0^\infty w'(t)w(t)dt \leq \frac{1}{\delta} \tag{2}$$

The problem considered in this paper regards the stabilization of system (1) through a dynamic output feedback compensator, described as follows:

$$\begin{array}{rcl} \dot{x}_c(t) & = & A_c x_c(t) + B_c y(t) + E_c(sat_{u_0}(y_c(t)) - y_c(t)) \\ & = & A_c x_c(t) + B_c sat_{y_0}(Cx(t)) + E_c(sat_{u_0}(y_c(t)) - y_c(t)) \\ y_c(t) & = & C_c x_c(t) + D_c y(t) \\ & = & C_c x_c(t) + D_c sat_{y_0}(Cx(t)) \end{array} \tag{3}$$

where $x_c \in \Re^{n_c}$ is the controller state, $y \in \Re^p$ is the controller input and $y_c \in \Re^m$ is the controller output. A_c, B_c, C_c, D_c and E_c are matrices of appropriate dimensions. E_c corresponds to a static anti-windup gain [11].

The connection between system (1) and the controller (3), leads to the following closed-loop system:

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + B sat_{u_0}(C_c x_c(t) + D_c sat_{y_0}(Cx(t))) + B_w w(t) \\ \dot{x}_c(t) & = & A_c x_c(t) + B_c sat_{y_0}(Cx(t)) + E_c(sat_{u_0}(y_c(t)) - y_c(t)) \\ y_c(t) & = & C_c x_c(t) + D_c sat_{y_0}(Cx(t)) \\ z(t) & = & C_z x(t) + D_z sat_{u_0}(C_c x_c(t) + D_c sat_{y_0}(Cx(t))) \end{array} \tag{4}$$

It is worth to notice that due to the sensor saturation, the resulting closed-loop system presents a nested saturation [29], [28].

Considering $w = 0$, the basin of attraction of system (4), denoted $\mathcal{B}_a$, is defined as the set of all $(x, x_c) \in \Re^n \times \Re^{n_c}$ such that for $(x(0), x_c(0)) \in \mathcal{B}_a$ the corresponding trajectory converges asymptotically to the origin. In particular, when the global stability of the system is ensured the basin of attraction corresponds to the whole state space.

In this paper, we focus on the following problems regarding system (4):

1. Tolerance and disturbance rejection ($w \neq 0$).
   The objective in this case consists in ensuring that the trajectories of the system are bounded for any disturbance satisfying (2) and, in addition, in providing an upper-bound for the $\mathcal{L}_2$-gain from the disturbance w to the regulated output z. In other words, we want to ensure input-to-state and input-to-output $\mathcal{L}_2$-stability.

2. Internal Stabilization.
   If $w(t) = 0, \forall t > t_1 \geq 0$, it should be ensured that the corresponding trajectories converge asymptotically to the origin. This means that for disturbances satisfying (2), the trajectories never leave the region of attraction of the closed-loop system. On the other hand, considering the free-disturbance case ($w = 0$), a relevant problem consists in designing the controller in order to maximize the region of attraction of the closed-loop system or, when possible, to ensure the global asymptotic stability of the origin. However, since in the general case the exact analytical characterization of the basin of attraction is not possible, we will be interested in maximizing estimates of the basin of attraction [17].

Regarding the problems above described, the design of the controller can be oriented in order to maximize the disturbance tolerance, the disturbance rejection or the region where the asymptotic stability of the closed-loop system is ensured.

3 Preliminaries

Let us define the following augmented vector

$$\xi = \begin{pmatrix} x \\ x_c \end{pmatrix} \in \Re^{n+n_c} \tag{5}$$

and the following matrices

$$\begin{array}{c} \mathbb{A} = \begin{pmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{pmatrix}; \ \mathbb{B}_u = \begin{pmatrix} B \\ E_c \end{pmatrix} \ \mathbb{B}_w = \begin{pmatrix} B_w \\ 0 \end{pmatrix} \\ \mathbb{B}_y = \begin{pmatrix} BD_c \\ B_c \end{pmatrix}; \ \mathbb{C}_y = \begin{pmatrix} C & 0 \end{pmatrix} \\ \mathbb{C} = \begin{pmatrix} D_cC & C_c \end{pmatrix}; \ \mathbb{D} = D_c \\ \mathbb{C}_z = \begin{pmatrix} C_z + D_zD_cC & D_zC_c \end{pmatrix}; \ \mathbb{D}_{zy} = D_zD_c \ ; \ \mathbb{D}_{zu} = D_z \end{array} \tag{6}$$

From (5) and (6), the closed-loop system can be written as:

$$\begin{array}{rcl} \dot{\xi}(t) & = & \mathbb{A}\xi(t) + \mathbb{B}_u\phi_{u_0}(t) + \mathbb{B}_y\phi_{y_0}(t) + \mathbb{B}_w w(t) \\ y_c(t) & = & \mathbb{C}\xi(t) + \mathbb{D}\phi_{y_0}(t) \\ z(t) & = & \mathbb{C}_z\xi(t) + \mathbb{D}_{zu}\phi_{u_0}(t) + \mathbb{D}_{zy}\phi_{y_0}(t) \end{array} \tag{7}$$

with nonlinearities ϕ_{y_0} and ϕ_{u_0} defined by

$$\begin{array}{rcl} \phi_{y_0}(t) & = & sat_{y_0}(\mathbb{C}_y\xi(t)) - \mathbb{C}_y\xi(t) \\ \phi_{u_0}(t) & = & sat_{u_0}(\mathbb{C}\xi(t) + \mathbb{D}\phi_{y_0}(t)) - (\mathbb{C}\xi(t) + \mathbb{D}\phi_{y_0}(t)) \end{array} \tag{8}$$

The nonlinearities ϕ_{y_0} and ϕ_{u_0} correspond to decentralized vector valued dead-zone functions. It should be noticed that ϕ_{u_0} depends on ϕ_{y_0}.

Consider now matrices $G \in \Re^{p\times(n+n_c)}$, $H_1 \in \Re^{m\times(n+n_c)}$ and $H_2 \in \Re^{m\times p}$ and define the following polyhedral sets:

$$S(y_0) = \{\xi \in \Re^{n+n_c}; -y_0 \preceq (\mathbb{C}_y - G)\xi \preceq y_0\} \tag{9}$$

$$S(u_0) = \{\xi \in \Re^{n+n_c}, \phi_{y0} \in \Re^p; -u_0 \preceq (\mathbb{C} - H_1)\xi + (\mathbb{D} - H_2)\phi_{y0} \preceq u_0\} \tag{10}$$

The nonlinearities ϕ_{y_0} and ϕ_{u_0} verify the following properties [11], [29].

Property 1 *If ξ belongs to $S(y_0)$, then*

$$\phi'_{y_0} T_y(\phi_{y_0} + G\xi) \leq 0 \tag{11}$$

for any diagonal positive definite matrix $T_y \in \Re^{p\times p}$.

Property 2 *If ξ and ϕ_{y_0} belong to $S(u_0)$, then*

$$\phi'_{u_0} T_u(\phi_{u_0} + H_1\xi + H_2\phi_{y_0}) \leq 0 \tag{12}$$

for any diagonal positive definite matrix $T_u \in \Re^{m\times m}$.

Define now an ellipsoidal set

$$\mathcal{E}(P, \delta^{-1}) = \{\xi \in \Re^{n+n_c}; \xi' P \xi \leq \delta^{-1}\} \tag{13}$$

with $P = P' > 0$ and $\delta > 0$.

Lemma 1 *If there exist a diagonal positive definite matrix $T_y \in \Re^{p\times p}$ such that the following linear matrix inequality are satisfied*

$$\begin{pmatrix} P & \mathbb{C}'_{y(i)} - G'_{(i)} \\ \bullet & \delta y^2_{0(i)} \end{pmatrix} \geq 0, \;\; i = 1, ..., p \tag{14}$$

$$\begin{pmatrix} P & G'T_y & \mathbb{C}'_{(i)} - H'_{1(i)} \\ \bullet & 2T_y & \mathbb{D}'_{(i)} - H'_{2(i)} \\ \bullet & \bullet & \delta u^2_{0(i)} \end{pmatrix} \geq 0, \;\; i = 1, ..., m \tag{15}$$

then the ellipsoidal set $\mathcal{E}(P, \delta^{-1})$ is contained in $S(u_0) \cap S(y_0)$.

Proof. It follows straightforwardly that (14) implies that $\mathcal{E}(P, \delta^{-1}) \subset S(y_0)$. In this case, from Property 1, it follows that $\mathcal{E}(P, \delta^{-1}) \subset S(u_0) \cap S(y_0)$, if

$$\begin{array}{c} \begin{pmatrix} \xi \\ \phi_{y_0} \end{pmatrix}' (\Omega - \begin{pmatrix} \Gamma'_{(i)} \\ \Theta'_{(i)} \end{pmatrix} \frac{1}{\delta u^2_{0(i)}} \begin{pmatrix} \Gamma_{(i)} & \Theta_{(i)} \end{pmatrix}) \begin{pmatrix} \xi \\ \phi_{y_0} \end{pmatrix} \geq 0, \; \forall i = 1, \ldots, m \\ \forall \xi, \phi_{y_0} \text{ such that } 2\phi'_{y_0} T_y(\phi_{y_0} + G\xi) \leq 0 \end{array} \tag{16}$$

where $\Omega = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}$, $\Gamma_{(i)} = \mathbb{C}_{(i)} - H_{1(i)}$ and $\Theta_{1(i)} = \mathbb{D}_{(i)} - H_{2(i)}$. Using the S-procedure and the Schur's complement, it follows that (16) is satisfied if (15) holds. ■

4 Main results

In this paper, only the design of a full order dynamic output controller is investigated. Hence, the results developed in the sequel hold for $n_c = n$, and therefore the state ξ of the closed-loop system (7) is an element of $\Re^{2n}$.

4.1 Local Stabilization

Theorem 1 *If there exist positive definite symmetric matrices* $X \in \Re^{n\times n}$, $Y \in \Re^{n\times n}$, *positive definite diagonal matrices* $S_u \in \Re^{m\times m}$, $S_y \in \Re^{p\times p}$ *and matrices* $Z \in \Re^{n\times n}$, $L \in \Re^{m\times n}$, $F \in \Re^{n\times p}$, $D \in \Re^{m\times p}$, $W_1 \in \Re^{m\times n}$, $W_2 \in \Re^{m\times n}$, $W_3 \in \Re^{p\times n}$, $W_4 \in \Re^{p\times n}$, $R \in \Re^{n\times m}$, $Q \in \Re^{m\times p}$ *and a scalar* γ *satisfying*

$$\begin{pmatrix} He\{AY+BL\} & Z & BS_u - W_1' & BDS_y - W_3' & B_w & YC_z' + L'D_z' \\ \bullet & He\{XA+FC\} & R - W_2' & FS_y - W_4' & XB_w & C_z' + C'D'D_z' \\ \bullet & \bullet & -2S_u & -Q & 0 & S_uD_z' \\ \bullet & \bullet & \bullet & -2S_y & 0 & S_yD'D_z' \\ \bullet & \bullet & \bullet & \bullet & -I_q & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\gamma I_r \end{pmatrix} < 0 \qquad (17)$$

$$\begin{pmatrix} Y & I_n & YC_{(i)}' - W_{3(i)}' \\ \bullet & X & C_{(i)}' - W_{4(i)}' \\ \bullet & \bullet & \delta y_{0(i)}^2 \end{pmatrix} \geq 0,\ i = 1, ..., p \qquad (18)$$

$$\begin{pmatrix} Y & I_n & W_3' & L_{(i)}' - W_{1(i)}' \\ \bullet & X & W_4' & C'D_{(i)}' - W_{2(i)}' \\ \bullet & \bullet & 2S_y & S_yD_{(i)}' - Q_{(i)}' \\ \bullet & \bullet & \bullet & \delta u_{0(i)}^2 \end{pmatrix} \geq 0,\ i = 1, ..., m \qquad (19)$$

then, considering $\xi(0) = 0$, *the controller (3) with*

$$\begin{array}{rcl} D_c & = & D \\ E_c & = & U^{-1}(R - XBS_u)S_u^{-1} \\ C_c & = & (L - D_cCY)(V')^{-1} \\ B_c & = & U^{-1}(F - XBD_c) \\ A_c & = & U^{-1}(Z' - (A + BD_cC)' - XAY - XBL - UB_cCY)(V')^{-1} \end{array} \qquad (20)$$

where matrices U *and* V *verify* $UV' = I_n - XY$, *is such that*

1. *when* $w \neq 0$

   a. *the closed-loop trajectories remain bounded in the set* $\mathcal{E}(P, \delta^{-1})$ *with*

   $$P = \begin{pmatrix} X & U \\ U' & \hat{X} \end{pmatrix} \qquad (21)$$

   b. $\|z\|_2^2 < \gamma\|w\|_2^2$.

2. *if* $w(t) = 0$, $\forall t > t_1 \geq 0$, $\xi(t)$ *converges asymptotically to the origin, i.e.* $\mathcal{E}(P, \delta^{-1})$ *is included in the basin of attraction of the closed-loop system (4) and it is a contractive set.*

Proof. Let $V(t) = \xi(t)'P\xi(t)$ be a candidate Lyapunov function and let $\dot{V}(t)$ be its time-derivative along system (7) (or (4)) trajectories. Define now $\mathcal{J}(t) = \dot{V}(t) - w'(t)w(t) + \frac{1}{\gamma}z'(t)z(t)$. If $\mathcal{J}(t) < 0$, one obtains that $\int_0^T \mathcal{J}(t)dt = V(T) - V(0) - \int_0^T w'(t)w(t)dt + \frac{1}{\gamma}\int_0^T z'(t)z(t)dt < 0$, $\forall\, T$. Hence, it follows that:

- since $\xi(0)$ is supposed to be zero, $V(0) = 0$ and $\xi(T)'P\xi(T) = V(T) < \|w\|_2^2 \leq \delta^{-1}$, $\forall T > 0$, i.e. the trajectories of the system do not leave the set $\mathcal{E}(P, \delta^{-1})$ for $w(t)$ satisfying (2);

- for $T \to \infty$, $\|z\|_2^2 < \gamma \|w\|_2^2$;
- if $w(t) = 0$, $\forall t > t_1 \geq 0$, then $\dot{V}(t) < -\frac{1}{\gamma} z'(t)z(t) < 0$, which ensures that $\xi(t) \to 0$ as $t \to \infty$.

Provided that $\xi(t) \in S(y_0)$ and $(\xi(t), \phi_{y_0}(t)) \in S(u_0)$ one obtains that

$$\mathcal{J}(t) \;\leq\; \mathcal{J}(t) - 2\phi'_{y_0}(t)T_y(\phi_{y_0}(t) + G\xi(t)) - 2\phi'_{u_0}(t)T_u(\phi_{u_0}(t) + H_1\xi(t) + H_2\phi_{y_0}(t)) = \mu(t)'\Xi\mu(t)$$

with $\mu(t) = \begin{pmatrix} \xi(t)' & \phi_{u_0}(t)' & \phi_{y_0}(t)' & w(t)' \end{pmatrix}'$ and

$$\Xi = \begin{pmatrix} \mathbb{A}'P + P\mathbb{A} & P\mathbb{B}_u - H_1'T_u & P\mathbb{B}_y - G'T_y & P\mathbb{B}_w \\ \bullet & -2T_u & -T_uH_2 & 0 \\ \bullet & \bullet & -2T_y & 0 \\ \bullet & \bullet & \bullet & -I_q \end{pmatrix} + \frac{1}{\gamma}\begin{pmatrix} \mathbb{C}_z' \\ \mathbb{D}_{zu}' \\ \mathbb{D}_{zy}' \\ 0 \end{pmatrix}\begin{pmatrix} \mathbb{C}_z & \mathbb{D}_{zu} & \mathbb{D}_{zy} & 0 \end{pmatrix} \quad (22)$$

Define

$$P = \begin{pmatrix} X & U \\ U' & \hat{X} \end{pmatrix};\; P^{-1} = \begin{pmatrix} Y & V \\ V' & \hat{Y} \end{pmatrix} \quad (23)$$

where $X \in \Re^{n\times n}$, $Y \in \Re^{n\times n}$, $\hat{X} \in \Re^{n\times n}$, $\hat{Y} \in \Re^{n\times n}$ are positive definite symmetric matrices and U, V matrices of appropriate dimensions. Thus, it follows:

$$\begin{array}{lcl} XY + UV' = I_n & ; & U'V + \hat{X}\hat{Y} = I_n \\ U'Y + \hat{X}V' = 0 & ; & XV + U\hat{Y} = 0 \end{array}$$

Define now the matrix [25]:

$$J = \begin{pmatrix} Y & V \\ I_n & 0 \end{pmatrix}$$

Note now that, from condition (18) and (19), it follows that $I_n - XY$ is nonsingular, which implies that is always possible to compute square and nonsingular matrices V and U verifying the equation $UV' = I_n - XY$. This fact ensures that J is nonsingular.

Applying Schur's complement in $\Xi < 0$, with Ξ given by (22), and pre and post-multiplying (22) respectively by $Diag(J; S_u; S_y; I_q; I_r)$ and $Diag(J'; S_u; S_y; I_q; I_r)$ with $S_y = T_y^{-1}$ and $S_u = T_u^{-1}$, one gets:

$$\begin{pmatrix} J\mathbb{A}'PJ' + JP\mathbb{A}J' & JP\mathbb{B}_uS_u - JH_1' & JP\mathbb{B}_yS_y - JG' & JP\mathbb{B}_w & J\mathbb{C}_z' \\ \bullet & -2S_u & -H_2S_y & 0 & S_u\mathbb{D}_{zu}' \\ \bullet & \bullet & -2S_y & 0 & S_y\mathbb{D}_{zy}' \\ \bullet & \bullet & \bullet & -I_q & 0 \\ \bullet & \bullet & \bullet & \bullet & -\gamma I_r \end{pmatrix} < 0 \quad (24)$$

Partitioning matrices H_1 and G as $H_1 = \begin{pmatrix} H_{11} & H_{12} \end{pmatrix}$ and $G = \begin{pmatrix} G_1 & G_2 \end{pmatrix}$, and considering the following change of variables

$$\begin{array}{c} L = D_cCY + C_cV',\; F = XBD_c + UB_c,\; M = VA_c'U', \\ Z = M + A + BD_cC + YA'X + L'B'X + YC'B_c'U', \\ R = XBS_u + UE_cS_u,\; D = D_c, \\ W_1 = H_{11}Y + H_{12}V',\; W_2 = H_{11} \\ W_3 = G_1Y + G_2V',\; W_4 = G_1, \end{array}$$

it follows that:

$$J\mathbb{A}'PJ' = \begin{pmatrix} YA' + L'B' & Z - (A + BD_cC) \\ (A + BD_cC)' & A'X + C'F' \end{pmatrix};\; JP\mathbb{B}_uS_u = \begin{pmatrix} BS_u \\ R \end{pmatrix};\; JP\mathbb{B}_yS_y = \begin{pmatrix} BDS_y \\ FS_y \end{pmatrix}$$

$$JH_1' = \begin{pmatrix} W_1' \\ W_2' \end{pmatrix} \; ; \; JG' = \begin{pmatrix} W_3' \\ W_4' \end{pmatrix} \; ; \; JP\mathbb{B}_w = \begin{pmatrix} B_w \\ XB_w \end{pmatrix} \; ; \; J\mathbb{C}_z' = \begin{pmatrix} YC_z' + L'D_z' \\ C_z' + C'D'D_z' \end{pmatrix} \; ; \; H_2 S_y = Q$$

$$S_u \mathbb{D}_{zu}' = S_u D_z' \; ; \; S_y \mathbb{D}_{zy}' = S_y D' D_z'$$

Hence, since J, S_y and S_u are nonsingular, it follows that if (17) is verified, $\mathcal{J}(t) < 0$ holds with the matrices A_c, B_c, C_c, D_c and E_c defined as in (20), provided that $\xi(t) \in S(y_0)$ and $(\xi(t), \phi_{y_0}(t)) \in S(u_0)$.

Consider now the set $\mathcal{E}(P, \delta^{-1})$, defined in (13). Pre and post-multiplying inequalities (15) respectively by $Diag(J; S_y; 1)$ and $Diag(J'; S_y; 1)$, pre and post-multiplying inequalities (14) respectively by $Diag(J; 1)$ and $Diag(J'; 1)$, and since

$$JPJ' = \begin{pmatrix} Y & I_n \\ I_n & X \end{pmatrix}, \; J\mathbb{C}' = \begin{pmatrix} L' \\ C'D' \end{pmatrix},$$

it follows, from Lemma 1, that the satisfaction of relations (18) and (19) ensures that $\mathcal{E}(P, \delta^{-1}) \subset S(u_0) \cap S(y_0)$. Thus, if relations (17), (18) and (19) are satisfied, one effectively obtains $\mathcal{J}(t) < 0$, $\forall \xi(t) \in \mathcal{E}(P, \delta^{-1})$, which completes the proof. ■

Remark 1 *In the case of a non-null initial condition $\xi(0)$, a positive scalar β has to be considered in order to ensure that the closed-loop trajectories remain bounded in $\mathcal{E}(P, \beta + \delta^{-1})$, $\forall \xi(0) \in \mathcal{E}(P, \beta)$. Hence, $\mathcal{E}(P, \beta)$ will be seen as a set of admissible initial conditions. From this, there clearly appears a trade-off between the size of the set of admissible conditions (given basically by β), the size of the region of stability (depending on $\beta + \delta^{-1}$) and the bound on the admissible disturbance (given by δ). Furthermore, the finite $\mathcal{L}_2$-gain from w to z will present a bias term and will read:*

$$\|z\|_2^2 \leq \gamma \|w\|_2^2 + \gamma \xi(0)' P \xi(0) \leq \gamma (\|w\|_2^2 + \beta).$$

A detailed discussion about this case can be found in [4], where the stabilization via state feedback of systems presenting only actuator saturation is considered.

4.2 Global stabilization

The results of the previous section can be adapted in order to provide global stabilizing conditions, applicable when the open-loop system is asymptotically stable. In this case, it can be ensured that the trajectories of the closed-loop system are bounded for any $w(t) \in \mathcal{L}_2$. Moreover, the origin of the system is ensured to be globally asymptotically stable.

Corollary 1 *If there exist positive definite symmetric matrices $X \in \Re^{n \times n}$, $Y \in \Re^{n \times n}$, positive definite diagonal matrices $S_u \in \Re^{m \times m}$, $S_y \in \Re^{p \times p}$ and matrices $Z \in \Re^{n \times n}$, $L \in \Re^{m \times n}$, $F \in \Re^{n \times p}$, $D \in \Re^{m \times p}$ and a scalar γ satisfying*

$$\begin{pmatrix} He\{AY + BL\} & Z & BS_u - L' & BDS_y - YC' & B_w & YC_z' + L'D_z' \\ \bullet & He\{XA + FC\} & R - C'D' & FS_y - C' & XB_w & C_z' + C'D'D_z' \\ \bullet & \bullet & -2S_u & -DS_y & 0 & S_u D_z' \\ \bullet & \bullet & \bullet & -2S_y & 0 & S_y D' D_z' \\ \bullet & \bullet & \bullet & \bullet & -I_q & 0 \\ \bullet & \bullet & \bullet & \bullet & \bullet & -\gamma I_r \end{pmatrix} < 0 \qquad (25)$$

$$\begin{pmatrix} Y & I \\ \bullet & X \end{pmatrix} > 0 \qquad (26)$$

then the controller (3) with

$$\begin{array}{rcl} D_c & = & D \\ E_c & = & U^{-1}(R - XBS_u)S_u^{-1} \\ C_c & = & (L - D_cCY)(V')^{-1} \\ B_c & = & U^{-1}(F - XBD_c) \\ A_c & = & U^{-1}(Z' - (A + BD_cC)' - XAY - XBL - UB_cCY)(V')^{-1} \end{array} \tag{27}$$

where matrices U and V verify $UV' = I_n - XY$, is such that the origin of the closed-loop system (4) is such that:

1. *when $w \neq 0$*
   a. *the closed-loop trajectories remain bounded for any $w(t) \in \mathcal{L}_2$ and any initial conditions.*
   b. *$\|z\|_2^2 < \gamma\|w\|_2^2 + \gamma\xi(0)'P\xi(0)$.*
2. *if $w(t) = 0$, $\forall t > t_1 \geq 0$, $\xi(t)$ converges asymptotically to the origin, i.e., the origin is globally asymptotically stable.*

Proof. Considering $H_1 = \mathbb{C}$, $H_2 = \mathbb{D} = D_c = D$ and $G = \mathbb{C}_y$, then it follows that:

$$\begin{array}{c} H_{11} = D_cC; H_{12} = C_c; G_1 = C; G_2 = 0 \\ W_1 = L; W_2 = DC; W_3 = CY; W_4 = C; Q = DS_y \end{array}$$

From this, the proof mimics the one of Theorem 1. Note that in this case the relations (11) and (12) are globally satisfied. ∎

5 Computational Issues

The conditions obtained in the previous section can be used to find the matrices of the dynamic controller (3) considering, for instance, the following optimization problems.

5.1 Maximization of the Disturbance Tolerance

The idea consists in maximizing the bound on the disturbance, for which we can ensure that the system trajectories remain bounded. This can be accomplished by the following optimization problem.

$$\begin{array}{c} \min \delta \\ \text{subject to} \;\; (17), (18) \text{ and } (19) \end{array} \tag{28}$$

Note that, in this case, we are not interested in the value of γ. Indeed, γ will assume a finite value to ensure that (17) is verified.

5.2 Maximization of the Disturbance Rejection

For an *a priori* given bound on the $\mathcal{L}_2$ norm of the admissible disturbances (given by $\frac{1}{\delta}$), the idea consists in minimizing the upper bound for the $\mathcal{L}_2$-gain of from $w(t)$ to $z(t)$. This can be obtained from the solution of the following optimization problem:

$$\begin{array}{c} \min \gamma \\ \text{subject to} \;\; (17), (18) \text{ and } (19) \text{ (or (25) and (26))} \end{array} \tag{29}$$

5.3 Maximization of the Region of Stability

We consider here the free-disturbance case, i.e. $w = 0$. The synthesis objective regards therefore the determination of a controller which leads to a stability region $\mathcal{E}(P, \delta^{-1})$ as large as possible, among all possible solutions for the inequalities of Theorem 1. Note that $\mathcal{E}(P, \delta^{-1})$ can be seen as an estimate of the region of attraction, and, in this case, the maximization of $\mathcal{E}(P, \delta^{-1})$ implicitly addresses the maximization of the region of attraction of the closed-loop system. Since $w = 0$, the relation (17) can be replaced by

$$\begin{pmatrix} He\{AY + BL\} & Z & BS_u - W_1' & BDS_y - W_3' \\ \bullet & He\{XA + FC\} & R - W_2' & FS_y - W_4' \\ \bullet & \bullet & -2S_u & -Q \\ \bullet & \bullet & \bullet & -2S_y \end{pmatrix} < 0 \tag{30}$$

which ensures that $\dot{V}(t) < 0$.

For a sake of simplicity, in this case, we set $\delta = 1$. The volume is a possible measure of how large $\mathcal{E}(P, 1)$ is [16]. This volume is proportional to $\det(P^{-1})$ with P^{-1} given in (23). A way to indirectly maximize the volume of the set $\mathcal{E}(P, 1)$ is to minimize the trace of P which can be written as

$$trace(P) = trace(X) + trace(\widehat{X})$$

From the definition of P and P^{-1}, we can deduce that

$$\widehat{X} = U'(X - Y^{-1})^{-1}U$$

and then minimize $trace(P)$ can be done by minimizing

$$trace(X) + \rho$$

where ρ is such that

$$\widehat{X} \leq \rho I_n$$

From the expression of $\widehat{X}$, the previous inequality is equivalent to

$$\begin{pmatrix} \rho I_n & U' & 0 \\ U & X & I_n \\ 0 & I_n & Y \end{pmatrix} \geq 0 \tag{31}$$

Note that in this last inequality, the matrix U appears. Following the steps of the proof of Theorem 1, we can note that matrix U is a degree of freedom related to the controller realization. The previous inequality connects matrices X, Y and U and its role is to select among all the possible solutions, the one enlarging the region of stability in the sense previously defined. In order to ensure that matrix U is nonsingular, we also consider the following constraint

$$U + U' > 0 \tag{32}$$

Hence, a solution to the problem of computing the controller in order to enlarge the basin of attraction of the closed-loop system can be indirectly addressed by solving the following optimization problem:

$$\begin{array}{c} \min \ trace(X) + \rho \\ \text{subject to (30), (18), (19), (31), (32)} \end{array} \tag{33}$$

Remark 2 *The optimization problems proposed above are not convex due to the products DS_y and FS_y appearing in relations (17) and (19). Note, however, that $S_y \in \Re^{p\times p}$ is a diagonal positive definite matrix. In the particular case of single-output systems ($p = 1$), S_y becomes a scalar and the optimal solution of (33) can be obtained from an iterative line search. For $p = 2$, a search for the optimal solution over a bi-dimensional grid (composed by the 2 elements of S_y) can be considered. For systems presenting high number of outputs a relaxation scheme, which translates the problem into a sequence of iterative LMI problems fixing S_y or (F, D) at each step, can be considered. In this case, the convergence of the procedure is always ensured, but not necessarily to the global optimal value. Furthermore, the convergence value will depend on the initialization of S_y or (F, D) in the iterative procedure. Hence, in all these cases an optimal or sub-optimal solution for (28), (29) or (33) can be easily obtained from the solution of LMI-based problems.*

6 Numerical examples

6.1 Example 1

Consider system (1) with the following matrices:

$$A = \begin{pmatrix} 0.5 & 0.7 \\ -0.7 & 0.5 \end{pmatrix}; \; B = B_w \begin{pmatrix} -1 \\ 0.5 \end{pmatrix}; \; C = C_z \begin{pmatrix} -1 & 1 \end{pmatrix}; \; D_z = 0$$

The saturation limits are defined by: $u_0 = 2$ and $y_0 = 2$. The open-loop system is exponentially unstable, then only regional (local) stabilization is possible.

Considering now problem (28), the optimal value of δ is 0.5988, which means that the maximal disturbance $\mathcal{L}_2$ bound for which it is possible, from the proposed conditions, to compute a controller that ensures the trajectories are bounded, is given by $1/\sqrt{\delta} = 1.2923$.

On the other hand, considering that the bound of the admissible disturbances is given by δ^{-1}, Table 1 shows the obtained values for γ solving problem (29) for different values of δ. Note that now, greater is δ (smaller is the admissible given disturbance bound), smaller is the upper bound on the $\mathcal{L}_2$ gain from w to z (i.e. higher is the disturbance rejection).

δ	γ
0.60	282.4406
0.61	80.6618
0.65	21.2647
0.70	11.3802
0.80	6.3783
0.90	4.8623
1.00	4.1695
10.00	2.4813

Table 1: Minimization of γ for a given δ

We consider now the problem of computing a controller that maximizes an estimate of the region of attraction for the closed-loop system. Thus, we consider the optimization problem (33). In order to avoid ill conditioning problems and also to ensure some time-domain performance when the systems operates in the linear region (i.e. when neither actuator nor sensor outputs are saturated), an LMI constraint is added to (33) in order to ensure that the eigenvalues of $\mathbb{A}$ are placed in the following strip in the complex plane [5]:

$$\mathcal{D} = \{s \in \mathcal{C}\ ;\ -\alpha_2 < s < -\alpha_1\ ; \alpha_1, \alpha_2 > 0\}$$

Thus, for $\alpha_1 = 0.05$, $\alpha_2 = 100$ and $S_y = 2.5$, we obtain the following stabilizing controller matrices:

$$A_c = \begin{pmatrix} -92.9775 & -8.1916 \\ -6.8976 & -75.4127 \end{pmatrix};\quad B_c = \begin{pmatrix} 505.2395 \\ 107.5444 \end{pmatrix}$$

$$C_c = \begin{pmatrix} 0.0565 & 0.0776 \end{pmatrix};\quad D_c = -1.3900;$$

$$E_c = \begin{pmatrix} -467.1186 & -437.2380 \end{pmatrix}'$$

In this case the domain of stability is given by:

$$P = \begin{pmatrix} 0.1566 & -0.1103 & 0.0003 & -0.0001 \\ -0.1103 & 0.6908 & 0.0000 & 0.0004 \\ 0.0003 & 0.0000 & 0.0001 & 0.0000 \\ -0.0001 & 0.0004 & 0.0000 & 0.0001 \end{pmatrix}$$

Simulation results for the initial condition $\xi(0) = \begin{pmatrix} -2.276 & 0.2731 & 0 & 0 \end{pmatrix}' \in \mathcal{E}(P, 1)$ is shown in figures 1 and 2. Note that both the actuator and sensor outputs saturate during the initial instants.

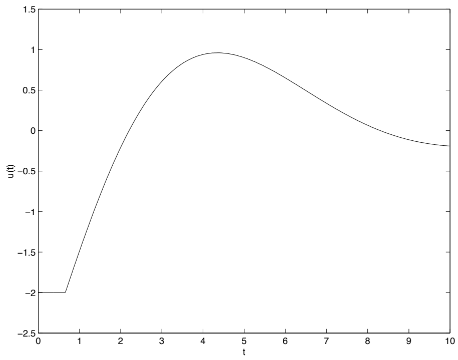

Figure 1: Actuator output

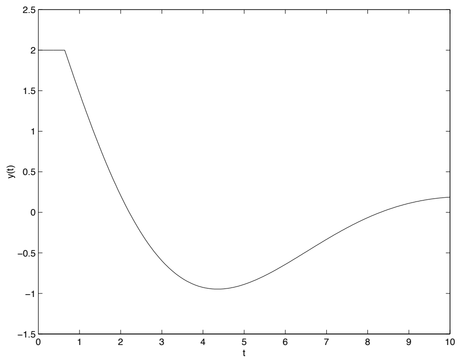

Figure 2: Sensor output

Table 2 shows the values of the optimization criterion, for different values of α_1. For each value, it is also shown the value of S_y for which the optimal solution to (33) has been obtained. Figure 3 illustrates, for each case in the table, the cut of the corresponding obtained sets $\mathcal{E}(P,1)$ in the plane defined by the states of the plant (i.e. for $x_c = 0$). The outer ellipsoid corresponds to $\alpha_1 = 0$, while the inner one corresponds to $\alpha_1 = 0.5$. This illustrates that more stringent is the time-domain performance (greater is α_1), smaller is the obtained region of stability.

α_1	$trace(P) + \rho$	S_y
0	0.7068	3.4
0.05	0.8477	2.5
0.1	1.0215	1.8
0.2	1.4910	0.9
0.3	2.4666	0.9
0.5	8.3144	0.5

Table 2: Trade-off: size of $\mathcal{E}(P,1)$ $\times$ performance constraint

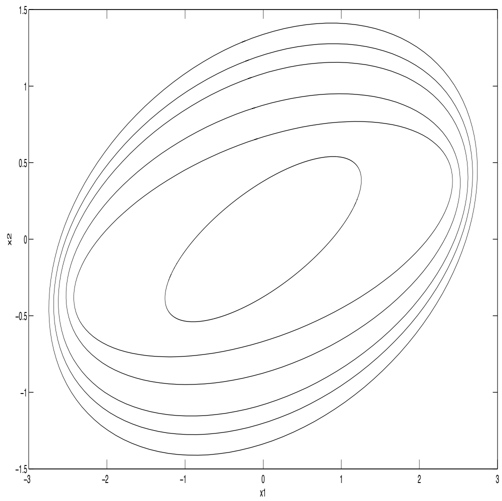

Figure 3: Stability domains $\times$ performance constraint

6.2 Example 2

Consider the longitudinal dynamics of an F-8 aircraft, given by the following matrices:

$$A = \begin{pmatrix} -0.8 & -0.006 & -12 & 0 \\ 0 & -0.014 & -16.64 & -32.2 \\ 1 & -0.0001 & -1.5 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; B = \begin{pmatrix} -19 & -3 \\ -0.66 & -0.5 \\ -0.16 & -0.5 \\ 0 & 0 \end{pmatrix}; B_w = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}; C_z = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}; D_z = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

This example has been addressed in [32], considering only the problem of actuator saturation (anti-windup problem) and in [3], considering only the sensor saturation problem. Henceforth, we address both problems simultaneously. We consider that the actuator and sensor bounds are, respectively, given by:

$$u_0 = \begin{pmatrix} 15 \\ 15 \end{pmatrix}; \; y_0 = \begin{pmatrix} 30 \\ 30 \end{pmatrix}$$

Considering that the admissible disturbances satisfy $||w||_2^2 \leq 2 \times 10^5$, i.e. $\delta = 5 \times 10^{-6}$, the solution of the optimization problem (29) with $S_y = Diag(0.05; 0.05)$ leads to $\gamma = 0.113225$ and the following controller[1]:

$$A_c = \begin{bmatrix} -1471.1687 & 92574.9022 & -573.676 & -1507.3994 \\ 13.2504 & -837.3562 & 5.1827 & 13.6311 \\ -7235.9778 & 458884.1667 & -2838.8208 & -7470.3309 \\ 4923.0174 & -311339.3013 & 1927.1434 & 5068.7017 \end{bmatrix}$$

$$B_c = \begin{bmatrix} 30.3083 & 9.942 \\ 0.08226 & 0.20071 \\ -73.7222 & -131.6338 \\ 3.609 & 52.5262 \end{bmatrix}$$

$$C_c = \begin{bmatrix} -2.2411 & 141.0032 & -0.87026 & -2.2904 \\ -164.5013 & 10388.2088 & -64.2876 & -169.091 \end{bmatrix}$$

$$D_c = \begin{bmatrix} 0.31812 & 0.23529 \\ 0.08495 & -1.5443 \end{bmatrix}$$

$$E_c = \begin{bmatrix} 81.9885 & 7.7072 \\ 0.43251 & -0.085127 \\ -210.2422 & 50.9469 \\ 26.8001 & -31.9154 \end{bmatrix}$$

Figures 4 and 5 show the responses of the closed-loop system to a disturbance defined as:

$$w_{(1)}(t) = \begin{cases} 316; & 0 \leq t < 2 \\ 0; & t \geq 2 \end{cases} \quad \text{and } w_{(2)}(t) = 0 \; \forall t \geq 0$$

It can be seen that the system outputs ($y(t) = z(t)$) as well as the controller outputs ($y_c(t)$) signals assume values larger than the saturation limits. This fact leads to an effective saturation of both sensors and actuators, but the designed controller stabilizes the system successfully.

We compare now with the behavior of a standard H_∞ controller computed disregarding the sensor and actuator bounds. This controller is given by the following matrices[2].

$$A_c = \begin{bmatrix} -116211.3388 & -338635.0467 & -1248882.079 & 486558.9343 \\ 109792.9383 & 234816.3287 & 944810.6231 & -375645.6678 \\ -271193.3741 & -673462.1362 & -2591901.7495 & 1020156.7694 \\ -647416.5118 & -1645980.3115 & -6293201.6411 & 2473151.1358 \end{bmatrix}$$

[1]For the computation of both controllers, we have chosen matrix $U = I$. In order to avoid conditioning problems, we have considered an additional pole placement constraints in order to have all the real parts of the closed-loop poles in $[-100; 0]$.

[2]We have considered $U = 10^4 I$ and pole placement constraints in order to have all the real parts of the closed-loop poles in $[-100; 0]$.

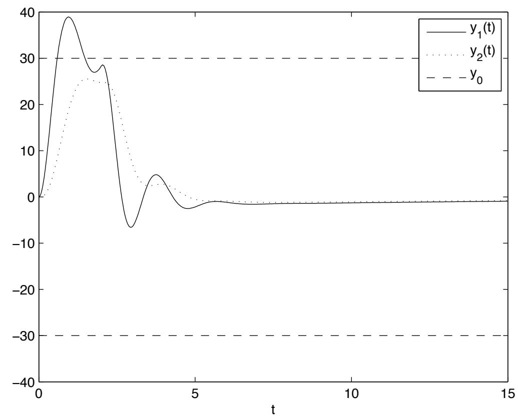

Figure 4: System outputs with the controller obtained from (29)

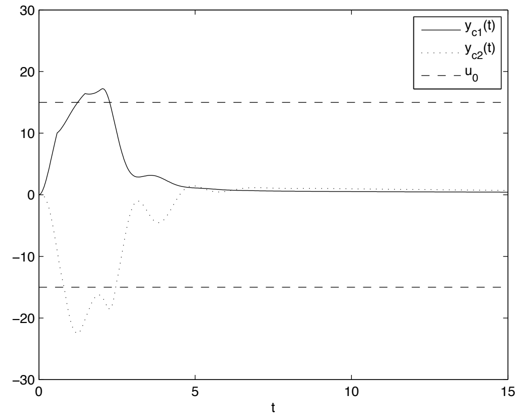

Figure 5: Controller outputs with the controller obtained from (29)

$$B_c = \begin{bmatrix} 9.2839 & -6.2083 \\ -6.403 & 4.5094 \\ 18.4158 & -12.6274 \\ 45.0275 & -30.7544 \end{bmatrix}$$

$$C_c = \begin{bmatrix} 255656.6981 & 49810.1907 & -148707.4599 & 96161.1536 \\ 0.0015527 & 4.6972e-05 & 0.0043181 & -0.0020795 \end{bmatrix}$$

$$D_c = \begin{bmatrix} 0 & 0 \\ -1 & 1.1609e-08 \end{bmatrix}$$

Figures 6 and 7 show respectively the closed-loop system outputs and the controller outputs obtained with the H_∞ controller when both saturations are present. As it can be seen, the behavior of the closed-loop system is in this case instable.

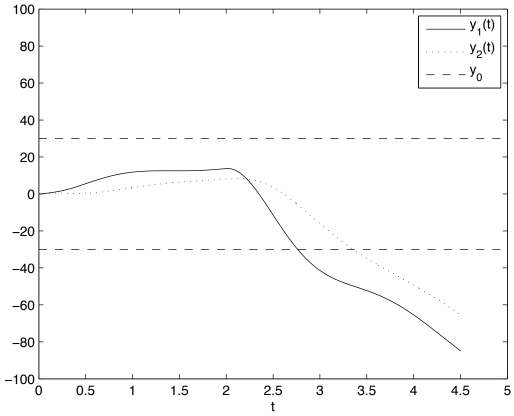

Figure 6: System outputs with the H_∞ controller

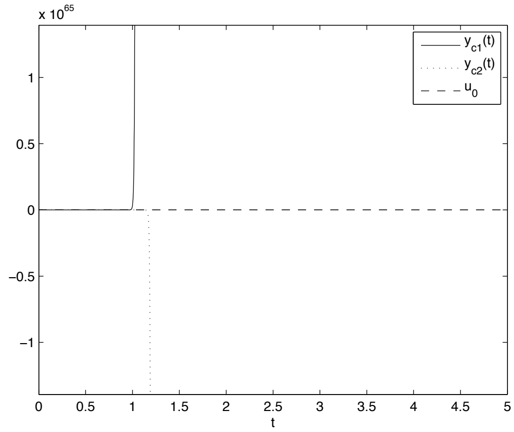

Figure 7: Controller outputs with the H_∞ controller

6.3 Example 3

Consider that system (1) is described by

$$A = \begin{pmatrix} -0.1 & 20 \\ -20 & -0.1 \end{pmatrix} \; ; \; B = B_w = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; ; \; C = C_z = \begin{pmatrix} 1 & 0 \end{pmatrix} \; ; \; D_z = 0$$

This system can represent, for instance, the transverse vibrational behavior of a cantilever beam, with one piezoelectric sensor and and one piezoelectric actuator, considering a reduced model containing only the fundamental frequency of oscillation. Due to the physical characteristics of the piezoelectric devices, they present voltage amplitude limitations when used as sensors as well as actuators.

Note that the open-loop system is in this case asymptotically stable. Then, the result proposed in section 4.2, concerning the global stabilization, can be applied.

In order to evaluate the control performances we consider the following two controllers[3]:

- The first controller is designed using the result proposed in Corollary 1 and the optimization problem (29), i.e., the actuator and sensor saturations are explicitly taken into account in the

[3]For the computation of both controllers, we have chosen $V = 1000I$ and to avoid conditioning problems, we have considered an additional pole placement constraints in order to have all the real parts of the closed-loop poles in $[-0.8; -0.15]$.

design. Hereafter, we denote this controller as $\mathbf{C_{sat}}$. The controller matrices are given by:

$$A_c = \begin{pmatrix} -0.50983 & 19.9998 \\ -20.0077 & -0.21359 \end{pmatrix} \;;\; B_c = \begin{pmatrix} 0.39066 \\ 0.001135 \end{pmatrix}$$

$$C_c = \begin{pmatrix} -0.0030157 & 4.8441e-05 \end{pmatrix} \;;\; D_c = -0.097788$$

$$E_c = \begin{pmatrix} 30.4653 \\ 0.4645 \end{pmatrix}$$

- The second controller is designed disregarding the saturations. As in the previous case, the performance measure considered was the linear $\mathcal{L}_2$ induced gain, i.e. it corresponds to a standard H_∞ dynamic controller (see for instance [25]). The controller presents the same structure and order as the previous controller, but with $E_c = 0$ (since no saturation is supposed to occur). Hereafter, we denote this controller as $\mathbf{C_{H\infty}}$. The controller matrices are given by:

$$A_c = \begin{pmatrix} -2.3675 & -308.5849 \\ 0.0060046 & 0.71646 \end{pmatrix} \;;\; B_c = \begin{pmatrix} 202735.3372 \\ -569.106 \end{pmatrix}$$

$$C_c = \begin{pmatrix} -0.033383 & -12.2323 \end{pmatrix} \;;\; D_c = 0$$

We consider that the following disturbance is applied to the system:

$$w(t) = \begin{cases} 10sin(17t); & 0 \leq t < \frac{20\pi}{17} \\ 0; & t \geq \frac{20\pi}{17} \end{cases}$$

Figure 8 shows the disturbance response of the closed-loop system with $\mathbf{C_{sat}}$ and $\mathbf{C_{H\infty}}$, considering that no saturation is present. As expected, in the absence of saturations, the controller $\mathbf{C_{H\infty}}$ performs much better.

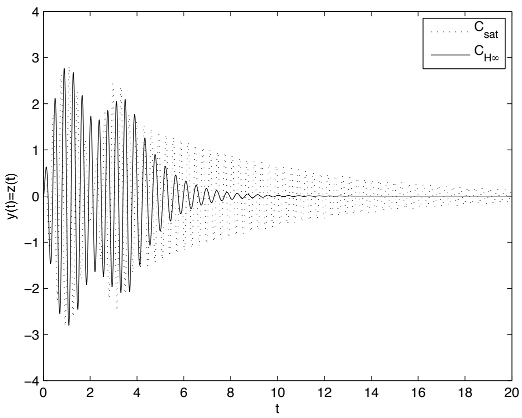

Figure 8: Regulated output $z(t)$ - no saturation considered

Now we compare the closed-loop responses of the two controllers in the presence of saturations. We consider that the sensor output saturate and the actuator output saturate at 1 (i.e. $u_0 = y_0 = 1$).

The Figures 9-12 show the closed-loop behavior with the controllers $\mathbf{C_{sat}}$ and $\mathbf{C_{H\infty}}$. Observer in Figure 9 that the regulated output with both controllers crosses the limit of 1 and then saturates. As it can be seen in Figure 10, the behavior of the closed-loop system with $\mathbf{C_{H\infty}}$ is less damped and slower

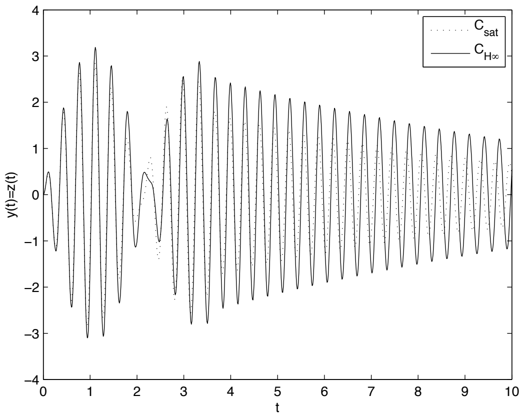

Figure 9: Regulated output $z(t)$ for $t = 0$ to 10.

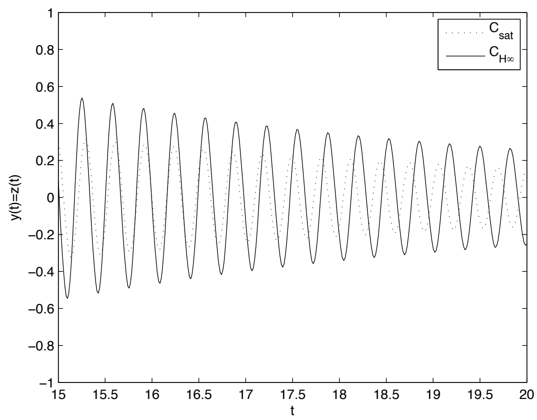

Figure 10: Zoom on the regulated output $z(t)$ for $t = 15$ to 20.

than the closed-loop system with controller $\mathbf{C_{sat}}$. Figures 11-12 show the controllers outputs ($y_c(t)$). Note that only the $\mathbf{C_{H\infty}}$ controller output crosses the limit of 0.5 and then saturates. Note also that the output of the controller $\mathbf{C_{sat}}$, shown in figure 11, presents a "pseudo-saturation" around 0.1. This happens because the system output $y(t) = z(t)$ actually saturates at ± 1 and the controller static gain is around 0.1.

Hence, when saturations are present, the performance achieved with the proposed controller $\mathbf{C_{sat}}$ is better than the one obtained with $\mathbf{C_{H\infty}}$, which has been designed by a standard H_∞ method without taking into account the possibility of sensor and actuator saturations.

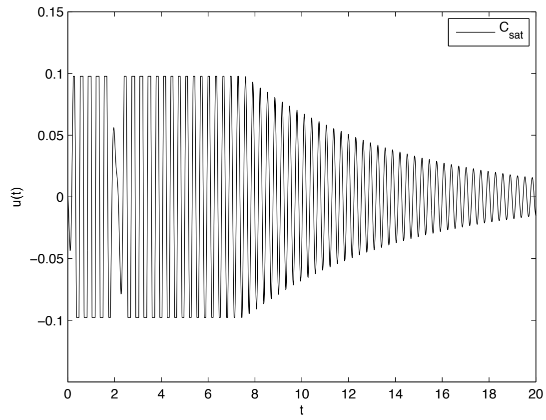

Figure 11: Controller output $y_c(t)$ - controller $\mathbf{C_{sat}}$.

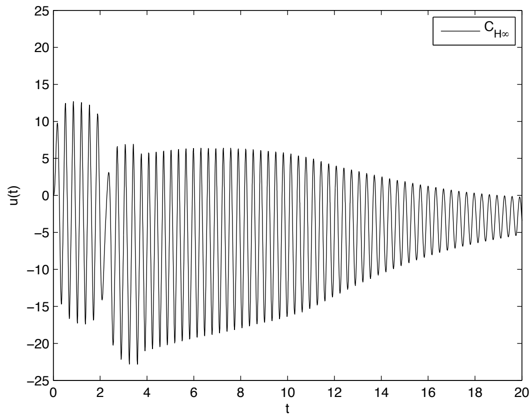

Figure 12: Controller output $y_c(t)$ - controller $\mathbf{C_{H\infty}}$.

7 Conclusion

In this paper, results concerning the problem of controlling linear systems with both sensors and actuators subject to saturation, through dynamic output feedback, have been presented. It has been shown that the closed-loop system in this case presents nested saturations. The effect of these nested saturations is taken into account by re-writing the system equations in order to transform the saturation nonlinearities in deadzone nonlinearities. The description of the saturations in the form of deadzone nonlinearities allows the application of some modified sector conditions. From the combination of these sector conditions and a quadratic approach, conditions for the synthesis of stabilizing output feedback dynamic controllers containing a static anti-windup loop have been proposed. The system being also subject to additive $\mathcal{L}_2$-bounded disturbance, the conditions ensure that the computed controller guarantees both internal asymptotic stability and finite $\mathcal{L}_2$-gain stability for the closed-loop system.

The obtained stabilization conditions have been expressed as "quasi" linear matrix inequalities. The weak nonlinearity which appears is due to the product of a diagonal multiplier (with order equal to the number of the inputs) and two other variables. Hence, LMI-based optimization problems have been proposed to compute the controller in order to optimize one of the following criteria: maximization of the $\mathcal{L}_2$ bound on the admissible disturbances (disturbance tolerance maximization); the minimization of the induced $\mathcal{L}_2$ gain between the disturbance and a regulated output (disturbance rejection maximization); or the maximization of the region where the asymptotic stability of the closed-loop system is ensured (maximization of the region of attraction).

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