# Dynamic output feedback stabilization for systems with sector-bounded nonlinearities and saturating actuators

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**Abstract**

In the present work a systematic methodology for computing dynamic output stabilizing feedback control laws for nonlinear systems subject to saturating inputs is presented. In particular, the class of Lur'e type nonlinear systems is considered. Based on absolute stability tools and a modified sector condition to take into account input saturation effects, an LMI framework is proposed to design the controller. Asymptotic as well as input-to-state and input-to-output (in a $\mathcal{L}_2$ sense) stabilizations are addressed both in regional (local) and global contexts. The controller structure is composed by a linear part, an anti-windup loop and a term associated to the output of the dynamic nonlinearity. Convex optimization problems are proposed to the controller synthesis considering different optimization criteria. A numerical example illustrates the potentialities of the methodology.

**Keywords:** Nonlinear control systems, Sector nonlinearities, Saturation, Dynamic output feedback, LMI.

## 1 Introduction

The design of most practical control systems requires to consider the presence of the nonlinearities that are inherent to the plant dynamics and/or to the physical actuator or sensor limitations, both in the analysis and in the synthesis phases. To cope with the presence of such nonlinearities, among the various existing nonlinear control systems approaches, absolute stability theory has been considered in the literature for analysis and synthesis of the so-called Lur'e systems ([1, 2]). More recently, the research on absolute stability has been intensified, mainly due to the possibility of using the Linear Matrix Inequality (LMI) framework and the existing related efficient numerical tools for computations [3].

Considering linear systems with saturating inputs, a large amount of works can be found in the literature. We can cite, for instance the following ones (see also references therein): [4], [5] and [6], considering state feedback control laws; [7], [8], [? ], [? ] and [? ] regarding linear dynamic output feedback controller synthesis; [9], [10] and [11] addressing the anti-windup synthesis. On the other hand, just few works deal with the problem of controlling nonlinear systems with saturating inputs in a systematic way. In the same line of the papers cited above, LMI conditions have also been proposed to synthesize stabilizing control laws for nonlinear systems subject to actuator amplitude limitations and for which the dynamics can be decomposed into the feedback interconnection of a linear system with a sector bounded nonlinearity: [12],[13] and [14] for precisely-known systems and [15] for some uncertain nonlinear systems. It should however be pointed out that, as in [16, 17], the considered control law consists of the feedback of the systems states and of the nonlinearity associated to the plant dynamics.

Our aim in the present work is to devise a systematic synthesis method to compute dynamic output feedback stabilizing controllers for Lur'e type nonlinear systems subject also to input saturation. The controller structure is composed by a linear compensator presenting the following inputs: the plant output, an anti-windup term (related to the input saturation) and the value of the plant sector bounded nonlinearity. Based on this structure, on the use of a quadratic Lyapunov function, and on sector conditions, LMI conditions to synthesize this kind of controllers in order to ensure asymptotic as well as input-to-state and input-to-output (considering $\mathcal{L}_2$ input disturbance signals) stabilizations are proposed. Both the local (regional) and the global stabilization cases are considered. Concerning the asymptotic stability, we will be interested in ensuring the local stability of the closed-loop system in a specific region or, provided some additional hypothesis are satisfied by the open-loop system, to ensure global asymptotic stability. In the input-to-state and input-to output cases, we will be interested in guaranteeing that the trajectories of the system remain bounded considering a $\mathcal{L}_2$ bound on the disturbance signal. ¿From the derived conditions convex optimization problems are proposed in order to compute the controller matrices aiming at: the maximization of the estimates of the basin of attraction of the closed-loop system, or the performance enhancement with a guaranteed region of stability; the maximization of the bound on the disturbance for which it is possible to ensure that the trajectories are bounded or, considering a given bound on the admissible disturbance, the minimization of the $\mathcal{L}_2$ gain with respect to a regulated output.

It should be pointed out that similar control problems have been studied in [18], where the authors separately consider a particular sector bounded nonlinearity (with dead-zone behavior) associated the dynamics of the plant or a static saturation nonlinearity, using in both cases a classical sector condition, which leads to BMI stabilization conditions. Differently from that work, our approach consider simultaneously a dynamic nonlinearity and a static input saturation. Furthermore, the proposed conditions will be stated directly in LMI form. Thus, the proposed results can be thought as an additional contribution for the treatment of more realistic nonlinear control systems, also in a more efficient computational way.

The paper is organized as follows. Section 2 presents the problem statement. Section 3 is concerned with the internal stabilization (i.e. asymptotic stabilization) while section 4 addresses the external stabilization (i.e input-to-state and input-to-output). Section 5 presents some convex optimization problems for the dynamic controller synthesis. Numerical examples are presented and commented in section 6. The paper finishes with some concluding remarks.

**Notations.** For two symmetric matrices, $A$ and $B$, $A > B$ means that $A - B$ is positive definite. $A'$ denotes the transpose of $A$ and $He\{A\} = A + A'$. $A_{(i)}$ and $A_{(i,j)}$ denote the i$^{th}$ row and the element $(i,j)$ of matrix $A$, respectively. $\star$ stands for symmetric blocks; $\bullet$ stands for an element that has no influence on the development. $I$ denotes an identity matrix of appropriate order. $diag\{A, B\}$ is the block-diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$.

## 2 Problem Statement

Consider a nonlinear continuous-time system represented by the Lur'e type system:

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) + Bu(t) + G\phi(z(t)) + B_w w(t) \\ y(t) & = & Cx(t) + D_w w(t) \\ z(t) & = & Lx(t) \end{array} \tag{1}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$ are the state and the control input, respectively, $w(t) \in \Re^r$ is the disturbance, $y(t) \in \Re^p$ correspond to the output and $z(t) \in \Re^q$ is the input to the nonlinear vector valued function (map) $\phi(\cdot) : \Re^q \to \Re^q$. $A$, $B$, $B_w$ $C$, $D_w$, $G$ and $L$ are real constant matrices of appropriate dimensions. The disturbance vector $w$ is assumed to be limited in energy, that is, $w(t) \in \mathcal{L}_2$ and for some scalar $\delta$, $0 \leq \frac{1}{\delta} < \infty$, it follows that:

$$\|w\|_2^2 = \int_0^\infty w(t)'w(t)dt \leq \frac{1}{\delta} \tag{2}$$

Regarding system (1) the following assumptions are considered:

**Assumption 1** *The nonlinearity $\phi(z)$ is continuous and verifies a cone bounded sector condition, i.e., $\phi(0) = 0$ and there exists a symmetric positive definite matrix $\Omega \in \Re^{q\times q}$ such that*

$$\phi(z)'\Delta\left(\phi(z) - \Omega z\right) \leq 0, \quad \forall z \in \tilde{\mathcal{S}}_1 \subseteq \Re^q \tag{3}$$

*where $\Delta \in \Re^{q\times q}$ is any diagonal matrix defined as follows:*

$$\Delta = diag(\delta_l), \quad \delta_l > 0, \ \forall l = 1, \ldots, q,$$

*The matrix $\Omega$ is supposed to be known. On the other hand, from the definition of $\Delta$, we see that (3) is verified if $q$ independent classical sector conditions $\phi_{(l)}(z)'(\phi_{(l)}(z)-\Omega_{(l)}z) \leq 0$, $\forall l = 1,\ldots,q$, are also verified. Thus, as it will be seen in the sequel, the matrix $\Delta$ will represent a degree of freedom in the controller design method ([15]). If $\tilde{\mathcal{S}}_1 \triangleq \Re^q$, then the sector condition (3) is globally verified, otherwise, it is only locally verified.*

**Assumption 2** *The system output $y(t)$ and the nonlinearity $\phi(z(t))$ are available for measurement.*

**Assumption 3** *The control inputs are supposed to be bounded as follows:*

$$-u_{0(i)} \leq u_{(i)} \leq u_{0(i)}, \qquad i = 1,\ldots,m \tag{4}$$

In consequence of the control bounds, the actual control signal to be injected in the system is a saturated one, i.e, considering the signal sent to the actuator given by $v(t)$, we have

$$u(t) = \quad sat(v(t)) \tag{5}$$

where each component of $sat(v)$ is defined, $\forall i = 1,...,m$, by: $sat(v)_{(i)} = sat(v_{(i)}) = sign(v_{(i)})\min(u_{0(i)}, |v_{(i)}|)$.

Consider now a nonlinear dynamic output feedback controller with the following structure:

$$\begin{aligned}\dot{x}_c(t) &= A_c x_c(t) + B_c u_c(t) + E_c\left(sat(v(t)) - v(t)\right) + G_c\phi(z(t))\\ y_c(t) &= C_c x_c(t) + D_c u_c(t) + F_c\phi(z(t))\end{aligned} \tag{6}$$

where $x_c(t) \in \Re^n$ is the controller state, $u_c(t)$ is the controller input and $y_c(t)$ is the controller output, matrices $A_c$, $B_c$, $C_c$, $D_c$, $E_c$, $F_c$ and $G_c$ are of appropriate dimensions. The term $E_c(sat(v(t)) - v(t))$ corresponds to a static anti-windup loop to mitigate the undesirable effects of windup caused by input saturation. The interconnection between the plant and the controller is given by: $v(t) = y_c(t)$, $u_c(t) = y(t)$.

Define now the following matrices:

$$\mathbb{A} = \begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix}, \mathbb{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \mathbb{R} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \mathbb{G} = \begin{bmatrix} G \\ 0 \end{bmatrix}, \mathbb{B}_w = \begin{bmatrix} B_w + BD_cD_w \\ B_cD_w \end{bmatrix},$$

$$\mathbb{K} = \begin{bmatrix} D_cC & C_c \end{bmatrix},\ \mathbb{L} = \begin{bmatrix} L & 0 \end{bmatrix} \text{ and } \mathbb{C} = \begin{bmatrix} C & 0 \end{bmatrix}.$$

Hence, considering an augmented state vector $\xi(t) = \begin{bmatrix} x(t)' & x_c(t)' \end{bmatrix}'$, the closed-loop system composed by the connection of the system (1) and the controller (6) reads:

$$\begin{aligned}\dot{\xi}(t) &= \mathbb{A}\xi(t) + (\mathbb{B}F_c + \mathbb{G} + \mathbb{R}G_c)\phi(z(t)) - (\mathbb{B} + \mathbb{R}E_c)\psi(y_c(t)) + \mathbb{B}_w w(t)\\ y(t) &= \mathbb{C}\xi(t) + D_w w(t)\\ z(t) &= \mathbb{L}\xi(t)\end{aligned} \tag{7}$$

where

$$\begin{aligned} y_c(t) = &\quad \mathbb{K}\xi(t) + F_c\phi(z(t)) + D_cD_ww(t) \\ \psi(y_c(t)) = &\quad y_c(t) - sat(y_c(t)) \end{aligned} \tag{8}$$

with $(\psi(y_c))_{(i)} \stackrel{\triangle}{=} y_{c(i)} - sat(y_c)_{(i)}, \quad i = 1, \ldots, m$. Note that, $\psi(y_c)$ corresponds to a decentralized deadzone nonlinearity.

In this paper, we focus on the following problems regarding the closed-loop system (7):

1. *Internal Stabilization.*

   Considering $w = 0$, the asymptotic stabilization of the closed-loop system should be ensured. In this sense, a relevant problem consists in designing the controller in order to maximize the region of attraction of the closed-loop system or, when possible, to ensure the global asymptotic stability of the origin. However, since in the general case the exact analytical characterization of the basin of attraction is not possible [1], we will be interested in maximizing estimates of the basin of attraction.

2. *External stabilization.*

   The objective in this case consists in ensuring that the trajectories of the system are bounded for any disturbance satisfying (2) and, in addition, in providing an upper-bound for the $\mathcal{L}_2$-gain from the disturbance $w$ to the output $y$. In other words, we want to ensure input-to-state and input-to-output stability. On the other hand, if $w(t) = 0$, $\forall t > t_1 \geq 0$, for some $t_1$, it should be ensured that the corresponding trajectories converge asymptotically to the origin. This means that for disturbances satisfying (2), the trajectories never leave the region of attraction of the closed-loop system.

Regarding the problems above described, the design of the controller can be oriented in order to maximize the region where the asymptotic stability of the closed-loop system is ensured, the disturbance tolerance or the disturbance rejection.

# 3 Internal Stabilization

In this section a condition to address the asymptotic stabilization problem is presented. In this case, we consider that the systems is disturbance free ($w(t) = 0$).

Before stating the main results, we recall a instrumental result to deal with deadzone nonlinearities. With this aim consider a matrix $H = \begin{bmatrix} H_\xi & H_\phi \end{bmatrix} \in \Re^{m\times(2n+q)}$ and define the set

$$\mathcal{S}_2 \stackrel{\triangle}{=} \left\{ \begin{bmatrix} \xi \\ \phi(z) \end{bmatrix} \in \Re^{2n+q}; |[(\mathbb{K} - H_\xi) \quad (F_c - H_\phi)]_{(i)} \begin{bmatrix} \xi \\ \phi(z) \end{bmatrix} | \leq u_{0(i)}, \quad i = 1, ..., m \right\}$$

Hence, the following Lemma, concerning the nonlinearity $\psi(y_c)$ can be stated.

**Lemma 1** *([11]) If* $\begin{bmatrix} \xi \\ \phi(z) \end{bmatrix} \in \mathcal{S}_2$ *then the relation*

$$\psi(y_c)'T\left(\psi(y_c) - H_\xi \xi - H_\phi \phi(z)\right) \leq 0 \tag{9}$$

*is verified for any matrix* $T \in \Re^{m\times m}$ *diagonal and positive definite.*

The inequality (9) can be seen as a *modified*(or *generalized*) sector condition, which hold specifically for deadzone nonlinearities. It can be shown (see [11]) that condition (9) encompasses the classical sector condition used, for instance in [18]. Furthermore, as it will be see in the sequel, in the case of regional stabilization, it allows to obtain conditions in a true LMI form.

### 3.1 Regional (Local) Stabilization

In this case we consider the set $\tilde{\mathcal{S}}_1$ defined as follows:

$$\tilde{\mathcal{S}}_1 \stackrel{\triangle}{=} \{z \in \Re^q \ ; \ |z_{(i)}| \leq \rho_{(i)}, \rho_{(i)} > 0, \ i = 1, \ldots, q\}.$$

Since $z(t) = Lx(t)$, it follows that $z(t) \in \tilde{\mathcal{S}}_1$ if and only if $\xi(t)$ belongs to the set

$$\mathcal{S}_1 = \{\xi \in \Re^{2n} \ ; \ |\mathbb{L}_{(i)}\xi| \leq \rho_{(i)}, \rho_{(i)} > 0, \ i = 1, \ldots, q\}.$$

**Theorem 1** *If there exist symmetric positive definite matrices* $X,\ Y \in \Re^{n\times n}$*, positive definite diagonal matrices* $S \in \Re^{m\times m}$*,* $S_\Delta \in \Re^{q\times q}$*, matrices* $\hat{A} \in \Re^{n\times n}$*,* $\hat{B} \in \Re^{n\times p}$*,* $\hat{C},\ \hat{H}_{\xi 1},\ \hat{H}_{\xi 2} \in \Re^{m\times n}$*,* $\hat{H}_\phi \in \Re^{m\times q}$*,* $\hat{D} \in \Re^{m\times p}$*,* $\hat{F} \in \Re^{m\times q}$*,* $\hat{G} \in \Re^{n\times q}$*, and a scalar* $\nu > 0$ *such that the following linear matrix inequalities are verified*

$$\begin{bmatrix} \mathbf{J_1} & A + \hat{A}' + B\hat{D}C & \mathbf{J_2} & -BS + \hat{H}'_{\xi 1} \\ \star & He\{YA + \hat{B}C\} & \mathbf{J_3} & \hat{E} + \hat{H}'_{\xi 2} \\ \star & \star & -2S_\Delta & \hat{H}'_\phi \\ \star & \star & \star & -2S \end{bmatrix} < 0 \tag{10}$$

$$\begin{bmatrix} X & \star & \star & \star \\ I & Y & \star & \star \\ \Omega LX & \Omega L & 2S_\Delta & \star \\ \hat{C}_{(i)} - \hat{H}_{\xi 1(i)} & \hat{D}_{(i)}C - \hat{H}_{\xi 2(i)} & \hat{F}_{(i)} - \hat{H}_{\phi(i)} & \nu\, u^2_{0(i)} \end{bmatrix} > 0 \quad i = 1, ..., m \tag{11}$$

$$\begin{bmatrix} X & \star & \star \\ I & Y & \star \\ L_{(i)}X & L_{(i)} & \nu\, \rho^2_{(i)} \end{bmatrix} > 0 \quad i = 1, ..., q \tag{12}$$

*where* $\mathbf{J_1} = He\{AX + B\hat{C}\}$, $\mathbf{J_2} = GS_\Delta + B\hat{F} + XL'\Omega$ *and* $\mathbf{J_3} = \hat{G} + L'\Omega$, *then the dynamic controller (6) with*

$$\begin{aligned}
A_c &= V^{-1}[\hat{A} - (YAX + YB\hat{C} + VB_cCX)](U')^{-1} \\
B_c &= V^{-1}(\hat{B} - YB\hat{D}) \\
C_c &= (\hat{C} - \hat{D}CX)(U')^{-1} \\
D_c &= \hat{D}, \quad E_c = -V^{-1}(\hat{E}S^{-1} + YB) \\
G_c &= -V^{-1}(-\hat{G}S_\Delta^{-1} + YG + YBF_c) \\
F_c &= \hat{F}S_\Delta^{-1}
\end{aligned} \tag{13}$$

*where matrices* $U$ *and* $V$ *verify* $VU' = I - YX$, *guarantees that the region* $\mathcal{E}(P, \nu^{-1}) = \{\xi \in \Re^{2n}; \xi' P \xi \leq \nu^{-1}\}$ *with* $P = \begin{bmatrix} Y & V \\ V' & \bullet \end{bmatrix}$ *is a domain of asymptotic stability for the closed-loop system (7).*

**Proof:** Define a candidate Lyapunov function $V(t) = \xi'(t)P\xi(t)$, with $P = \begin{bmatrix} Y & V \\ V' & \bullet \end{bmatrix}$ and $P^{-1} = \begin{bmatrix} X & U \\ U' & \bullet \end{bmatrix}$. It follows that:

$$\dot{V}(t) = 2\xi'(t)\mathbb{A}'P\xi(t) - 2\psi'(y_c(t))(\mathbb{B} + \mathbb{R}E_c)'P\xi(t) + 2\phi'(z(t))(\mathbb{B}F_c + \mathbb{G} + \mathbb{R}G_c)'P\xi(t) \tag{14}$$

Suppose now that

$$\dot{V}(t) - 2\psi(y_c(t))'T(\psi(y_c(t)) - H_\xi\xi(t) - H_\phi\phi(z(t))) - 2\phi(z(t))'\Delta(\phi(z(t)) - \Omega\mathbb{L}\xi(t)) < 0 \tag{15}$$

Hence, from Lemma 1 and Assumption 1, provided that $\xi(t) \in \mathcal{S}_1 \cap \mathcal{S}_2$, it follows that $\dot{V}(t) < 0$.

The expression (15) is equivalent to $\eta(t)'\Gamma\eta(t) < 0$ with:

$$\eta(t) = \begin{bmatrix} \xi(t) \\ \phi(z(t)) \\ \psi(y_c(t)) \end{bmatrix} \text{ and } \Gamma = \begin{bmatrix} He\{\mathbb{A}'P\} & P\mathbf{J_4} + \mathbb{L}'\Omega\Delta & -P\mathbf{J_5} + H_\xi'T \\ \star & -2\Delta & H_\phi'T \\ \star & \star & -2T \end{bmatrix} \tag{16}$$

where $\mathbf{J_4} = \mathbb{G} + \mathbb{R}G_c + \mathbb{B}F_c$ and $\mathbf{J_5} = \mathbb{B} + \mathbb{R}E_c$.

Next we show that (15) is equivalent to $\Gamma < 0$ and then it assures that $\dot{V}(t) < 0$, provided $\xi(t) \in \mathcal{S}_1 \cap \mathcal{S}_2$. With this aim, define a matrix $\Pi = \begin{bmatrix} X & I \\ U' & 0 \end{bmatrix}$ ([19]). Note that, from condition (11), it follows that $I - YX$ is nonsingular, which implies that is always possible to compute square and nonsingular matrices $V$ and $U$ such that the equation $VU' = I - YX$ is verified. This fact ensures that $\Pi$ is nonsingular.

Defining $S_\Delta = \Delta^{-1}$ and $S = T^{-1}$ and pre and post-multiplying matrix $\Gamma$ by $Diag(\Pi', S_\Delta', S')$ and $Diag(\Pi, S_\Delta, S)$ respectively, the following matrix is obtained

$$\begin{bmatrix} He\{\Pi'\mathbb{A}'P\Pi\} & \Pi'P\mathbf{J_4}S_\Delta + \Pi'\mathbb{L}'\Omega & -\Pi'P\mathbf{J_5}S + \Pi'H_\xi' \\ \star & -2S_\Delta & S_\Delta H_\phi' \\ \star & \star & -2S \end{bmatrix}. \tag{17}$$

Considering now the following change of variables:

$$
\begin{aligned}
&\hat{A} = YAX + YBD_cCX + VB_cCX + YBC_cU' + VA_cU',\\
&\hat{B} = YBD_c + VB_c, \quad \hat{C} = C_cU' + D_cCX, \quad \hat{D} = D_c,\\
&\hat{E} = -(YBS + VE_cS), \quad \hat{F} = F_cS_\Delta,\\
&\hat{G} = YGS_\Delta + VG_cS_\Delta + YBF_cS_\Delta, \quad \hat{H}_\phi = H_\phi S_\Delta\\
&\hat{H}_{\xi 1} = H_{\xi 1}X + H_{\xi 2}U', \quad \hat{H}_{\xi 2} = H_{\xi 1}
\end{aligned}
$$

it follows that

$$
\begin{array}{l}
\Pi' P \Pi = \begin{bmatrix} X & I \\ I & Y \end{bmatrix}; \ \Pi' P \mathbf{J_5} S = \begin{bmatrix} BS \\ -\hat{E} \end{bmatrix}; \ \Pi' P \mathbf{J_4} S_\Delta = \begin{bmatrix} GS_\Delta + B\hat{F} \\ \hat{G} \end{bmatrix}; \\
\Pi' H'_\xi = \begin{bmatrix} \hat{H}'_{\xi 1} \\ \hat{H}'_{\xi 2} \end{bmatrix}; \ \Pi' P \mathbb{A} \Pi = \begin{bmatrix} AX + B\hat{C} & A + B\hat{D}C \\ \hat{A} & YA + \hat{B}C \end{bmatrix}.
\end{array}
\tag{18}
$$

Hence, since $\Pi$, $S_\Delta$ and $S$ are nonsingular, it follows that (10) is equivalent to $\Gamma < 0$, which, from (15), implies that $\dot{V}(t) < 0$ holds with the matrices $A_c$, $B_c$, $C_c$, $D_c$, $E_c$, $F_c$ and $G_c$ defined as in (13), provided $\xi(t) \in \mathcal{S}_1 \cap \mathcal{S}_2$.

Consider now $\mathcal{E}(P, \nu^{-1})$. Pre and post-multiplying inequalities (11) respectively by $\begin{bmatrix} (\Pi^{-1})' & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and its transpose, and since $\mathbb{K}\Pi = [D_cCX + C_cU' \quad D_cC] = [\hat{C} \quad \hat{D}C]$, it follows that condition (11) ensures that $\mathcal{E}(P, \nu^{-1}) \subset \mathcal{S}_2$ (see [12]). By similar reasoning, (12) implies that $\mathcal{E}(P, \nu^{-1}) \subset \mathcal{S}_1$. Thus, if relations (10),(11),(12) are verified, one effectively obtains $\dot{V}(t) < 0$, $\forall \xi \in \mathcal{E}(P, \nu^{-1})$, which concludes the proof. □

**Remark 1** *The result of the Theorem can be straightforwardly extended to treat the local stabilization when the nonlinearity $\phi(z(t))$ globally satisfies the sector condition (3), i.e., when $\tilde{\mathcal{S}}_1 \triangleq \Re^q$. For this, it suffices to consider $H_\phi = F_c$ in condition (10), to eliminate the third row and third column matrices in (11) and to eliminate (12).*

## 3.2 Global Stabilization

In this case we consider that the open-loop matrix $A$ is Hurwitz and the set $\tilde{\mathcal{S}}_1 \triangleq \Re^q$, i.e. the nonlinearity $\phi(z)$ is such that the sector condition (3) is globally verified.

**Corollary 1** *If there exist symmetric positive definite matrices $X$, $Y \in \Re^{n\times n}$, positive definite diagonal matrices $S \in \Re^{m\times m}$, $S_\Delta \in \Re^{q\times q}$ and matrices $\hat{A} \in \Re^{n\times n}$, $\hat{B} \in \Re^{n\times p}$, $\hat{C}$, $\hat{D} \in \Re^{m\times p}$, $\hat{F} \in \Re^{m\times q}$ and*

$\hat{G} \in \Re^{n\times q}$ *such that the following linear matrix inequalities are verified*

$$\begin{bmatrix} \mathbf{J_1} & A + \hat{A}' + B\hat{D}C & \mathbf{J_2} & -BS + \hat{C}' \\ \star & He\{YA + \hat{B}C\} & \mathbf{J_3} & \hat{E} + C'\hat{D} \\ \star & \star & -2S_\Delta & \hat{F}' \\ \star & \star & \star & -2S \end{bmatrix} < 0, \tag{19}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0 \tag{20}$$

*then the dynamic controller (6) with the matrices defined as in (13), where matrices $U$ and $V$ verify $VU' = I - YX$, guarantees the global asymptotic stability of the origin of the closed-loop system (7).*

**Proof:** Consider $H_\xi = \mathbb{K}$ and $H_\phi = F_c$. It follows that the sector condition

$$\psi(y_c(t))'T\left(\psi(y_c(t)) - \mathbb{K}\xi(t) - F_c\phi(z(t))\right) \leq 0$$

is verified for all $\xi(t) \in \Re^{2n}$ and $\phi(z(t)) \in \Re^q$ . In this case, it is easy to see that (19) corresponds to (10) and the global asymptotic stability of the origin follows. Inequality (20) ensures that $I - YX$ is nonsingular. □

# 4 External Stabilization

In this section we focus on the input-to-state and input-to-output stabilizations. Considering that $w(t)$ satisfies (2), we provide conditions to compute a controller in order to ensure that the state trajectories are bounded and, in addition, to ensure finite $\mathcal{L}_2$ gain between the disturbance input to the regulated output. I this case, we assume that $\xi(0) = 0$.

Since $w(t)$ is now supposed to be different from zero, a slightly modified version of Lemma 1 has to be considered as follows.

**Lemma 2** *If* $\begin{bmatrix} \xi \\ \phi(z) \end{bmatrix} \in \mathcal{S}_2$ *then the relation*

$$\psi(y_c)'T\left(\psi(y_c) - H_\xi\xi - H_\phi\phi(z) - D_cD_w w\right) \leq 0 \tag{21}$$

*is verified for any matrix $T \in \Re^{m\times m}$ diagonal and positive definite.*

## 4.1 Regional Stabilization

**Theorem 2** *Suppose that $||w||_2^2 \leq \delta^{-1}$ and $\xi(0) = 0$. If there exist symmetric positive definite matrices $X,\ Y \in \Re^{n\times n}$, positive definite diagonal matrices $S \in \Re^{m\times m}$, $S_\Delta \in \Re^{q\times q}$, matrices $\hat{A} \in \Re^{n\times n}$, $\hat{B} \in \Re^{n\times p}$,*

*$\hat{C}$, $\hat{H}_{\xi 1}$, $\hat{H}_{\xi 2} \in \Re^{m\times n}$, $\hat{H}_{\phi} \in \Re^{m\times q}$, $\hat{D} \in \Re^{m\times p}$, $\hat{F} \in \Re^{m\times q}$, $\hat{G} \in \Re^{n\times q}$, and scalars $\delta > 0$ and $\gamma > 0$ such that the following linear matrix inequalities are verified*

$$\begin{bmatrix} \mathbf{J_1} & A+\hat{A}'+B\hat{D}C & \mathbf{J_2} & -BS+\hat{H}'_{\xi 1} & B_w+B\hat{D}D_w & XC' \\ \star & He\{YA+\hat{B}C\} & \mathbf{J_3} & \hat{E}+\hat{H}'_{\xi 2} & YB_w+\hat{B}D_w & C' \\ \star & \star & -2S_\Delta & \hat{H}'_\phi & 0 & 0 \\ \star & \star & \star & -2S & \hat{D}D_w & 0 \\ \star & \star & \star & \star & -I_r & D'_w \\ \star & \star & \star & \star & \star & -\gamma I_q \end{bmatrix} < 0 \qquad (22)$$

$$\begin{bmatrix} X & \star & \star & \star \\ I & Y & \star & \star \\ \Omega L X & \Omega L & 2S_\Delta & \star \\ \hat{C}_{(i)}-\hat{H}_{\xi 1(i)} & \hat{D}_{(i)}C-\hat{H}_{\xi 2(i)} & \hat{F}_{(i)}-\hat{H}_{\phi(i)} & \delta\, u^2_{0(i)} \end{bmatrix} > 0 \quad i=1,...,m \qquad (23)$$

$$\begin{bmatrix} X & \star & \star \\ I & Y & \star \\ L_{(i)}X & L_{(i)} & \delta\rho^2_{(i)} \end{bmatrix} > 0 \quad i=1,...,q \qquad (24)$$

*where $\mathbf{J_1} = He\{AX + B\hat{C}\}$, $\mathbf{J_2} = GS_\Delta + B\hat{F} + XL'\Omega$ and $\mathbf{J_3} = \hat{G} + L'\Omega$, then the dynamic controller (6) with matrices computed as in (13) where matrices $U$ and $V$ verify $VU' = I - YX$, is such that*

*a. the closed-loop trajectories remain bounded in the set $\mathcal{E}(P, \delta^{-1})$ with*

$$P = \begin{pmatrix} Y & V \\ V' & \bullet \end{pmatrix} \qquad (25)$$

*b. $\|y\|_2^2 < \gamma\|w\|_2^2$.*

*c. if there exist $t_1$ such that $w(t) = 0$, $\forall t > t_1 \geq 0$, $\xi(t)$ converges asymptotically to the origin, i.e. $\mathcal{E}(P, \delta^{-1})$ is included in the basin of attraction of the closed-loop system and it is a contractive set.*

**Proof.** Let $V(t) = \xi(t)'P\xi(t)$ be a candidate Lyapunov function and let $\dot{V}(t)$ be its time-derivative along system (7) trajectories. Define now $\mathcal{J}(t) = \dot{V}(t) - w'(t)w(t) + \frac{1}{\gamma}y'(t)y(t)$. Using the same similarity transformation applied in the proof of Theorem 1, the relation (22) implies that $\mathcal{J}(t) < 0$, provided $\xi(t) \in \mathcal{S}_1 \cap \mathcal{S}_2$. Thus, in this case, one obtains that $\int_0^T \mathcal{J}(t)dt = V(T) - V(0) - \int_0^T w'(t)w(t)dt + \frac{1}{\gamma}\int_0^T y'(t)y(t)dt < 0$, $\forall\, T$. Hence, it follows that:

- since $\xi(0)$ is supposed to be zero, $V(0) = 0$ and $\xi(T)'P\xi(T) = V(T) < \|w\|_2^2 \leq \delta^{-1}$, $\forall T > 0$, i.e. the trajectories of the system do not leave the set $\mathcal{E}(P, \delta^{-1})$ for $w(t)$ satisfying (2);
- for $T \to \infty$, $\|y\|_2^2 < \gamma\|w\|_2^2$;

- if $w(t) = 0$, $\forall t > t_1 \geq 0$, then $\dot{V}(t) < -\frac{1}{\gamma}y'(t)y(t) < 0$, which ensures that $\xi(t) \to 0$ as $t \to \infty$.

On the other hand, LMIs (23) and (24) ensure that $\mathcal{E}(P, \delta^{-1}) \subset \mathcal{S}_1 \cap \mathcal{S}_2$. Hence, this fact together with the satisfaction of (22) ensure that the trajectories effectively never leave the set $\mathcal{E}(P, \delta^{-1})$ which concludes the proof. ■

## 4.2 Global Stabilization

As in section 3.2, in this case we consider the set $\tilde{\mathcal{S}}_1 \triangleq \Re^q$ and the open-loop matrix $A$ is Hurwitz.

**Corollary 2** *If there exist symmetric positive definite matrices $X$, $Y \in \Re^{n\times n}$, positive definite diagonal matrices $S \in \Re^{m\times m}$, $S_\Delta \in \Re^{q\times q}$ and matrices $\hat{A} \in \Re^{n\times n}$, $\hat{B} \in \Re^{n\times p}$, $\hat{C}$, $\hat{D} \in \Re^{m\times p}$, $\hat{F} \in \Re^{m\times q}$ and $\hat{G} \in \Re^{n\times q}$ such that the following linear matrix inequalities are verified*

$$\begin{bmatrix} \mathbf{J_1} & A + \hat{A}' + B\hat{D}C & \mathbf{J_2} & -BS + \hat{C}' & B_w + B\hat{D}D_w & XC' \\ \star & He\{YA + \hat{B}C\} & \mathbf{J_3} & \hat{E} + C'\hat{D} & YB_w + \hat{B}D_w & C' \\ \star & \star & -2S_\Delta & \hat{F}' & 0 & 0 \\ \star & \star & \star & -2S & \hat{D}D_w & 0 \\ \star & \star & \star & \star & -I_r & D_w' \\ \star & \star & \star & \star & \star & -\gamma I_q \end{bmatrix} < 0 \tag{26}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} > 0 \tag{27}$$

*then the dynamic controller (6) with the matrices defined as in (13), where matrices $U$ and $V$ verify $VU' = I - YX$, guarantees that*

*a. the closed-loop trajectories remain bounded for any any $w(t) \in \mathcal{L}_2$.*

*b. $\|z\|_2^2 < \gamma\|w\|_2^2$.*

*c. if there exist $t_1$ such that $w(t) = 0$, $\forall t \geq t_1 \geq 0$, $\xi(t)$ converges asymptotically to the origin.*

**Proof:** As in the proof of Corollary 1, consider $H_\xi = \mathbb{K}$ and $H_\phi = F_c$. It follows that the sector condition

$$\psi(y_c(t))'T\left(\psi(y_c(t)) - \mathbb{K}\xi(t) - F_c\phi(z(t)) - D_cD_ww\right) \leq 0$$

is verified for all $\xi(t) \in \Re^{2n}$ and $\phi(z(t)) \in \Re^q$ . In this case, it is easy to see that (26) corresponds to (22) and the global asymptotic stability of the origin follows.□

**Remark 2** *In addition to the fact of introducing additional degrees of freedom in the synthesis problem, another important role of the anti-windup term in the controller (6) is that it allows to convexify the problem, i.e. to obtain true LMI conditions. Note that if $E_c$ is not considered, it is not be possible to eliminate the nonlinear term $YGS_\Delta$ using an auxiliary variable $\hat{G}$.*

**Remark 3** *The same comments done in Remark 2 for the anti-windup term apply for the presence of the nonlinearity* $\phi(z)$ *in the controller. Indeed, without* $G_c$ *and/or* $F_c$*, the nonlinear term* $YGS_\Delta$ *would last in the conditions. Hence, if the nonlinearity* $\phi(z)$ *cannot be measured or evaluated, a BMI condition have to be considered. Note however that, since* $S_\Delta$ *is a diagonal matrix, the bilinearity is not too severe. In particular, if* $q = 1$ *(which is in general the case) or* $q = 2$*, the solution to the BMI can be easily found solving LMI problems in a unidimensional or a bidimensional grid on the variable* $S_\Delta$*. In the generic case, some LMI iterative procedures, where either* $Y$ *or* $S_\Delta$*, are fixed in each step can be considered in order to determine the dynamic controller.*

# 5 Optimization Problems

## 5.1 Enlargement of the stability region

An implicit objective in the synthesis of the stabilizing controller (6) can be the maximization of the region (basin) of attraction of the closed-loop system. Nevertheless, the exact characterization of this region is in general not possible and it depends directly on the controller that is being synthesized. The idea is therefore to compute a controller leading to a maximized estimate of the region of attraction. In other words, we want to compute (6) such that the associated region of asymptotic stability $\mathcal{E}(P, \nu^{-1})$ is as large as possible considering some size criterion. This can be addressed, for instance, if we consider a polyhedral set $\Xi$ described by the convex hull of its vertices:

$$\Xi \stackrel{\triangle}{=} Co\{v_1, v_2, \ldots, v_{n_r}\},\ v_l \in \Re^{2n},\ l = 1, \ldots, n_r$$

and a scaling factor $\beta$. Hence, recalling Theorem 1, we aim at searching for matrices $X, Y, \hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{E}, \hat{F}, \hat{G}, \hat{H}_\xi$ and $\hat{H}_\phi$ in order to obtain $\beta\ \Xi \subset \mathcal{E}(P, \nu^{-1})$ with $\beta$ as large as possible.

The vectors $v_l$ can in fact be viewed as directions in which we want to maximize the region of attraction. Considering that $\beta v_l \in \mathcal{E}(P, \nu^{-1})$ is equivalent to

$$\beta v_l' P v_l \beta \leq \nu^{-1} \tag{28}$$

and considering $\eta = 1/\beta^2$, it follows that the maximization of the ellipsoid $\mathcal{E}(P, \nu^{-1})$ along the directions $v_l$ is equivalent to the minimization of $\eta$. Hence, for a given value $\nu > 0$, $\mathcal{E}(P, \nu^{-1})$ can be maximized along the directions given by generic vectors $v_l = \begin{bmatrix} v_{l1}' & v_{l2}' \end{bmatrix}'$ where $v_{l1} \in \Re^n$ and $v_{l2} \in \Re^n$, by solving the following convex optimization problem:

$$\min_{V,\eta} \eta$$

subject to

$$(i) \begin{bmatrix} \nu^{-1}\eta & v'_{l1} & v'_{l1}Y + v'_{l2}V' \\ v_{l1} & X & I \\ Yv_{l1} + Vv_{l2} & I & Y \end{bmatrix} \geq 0 \tag{29}$$

$$l = 1, \ldots, r$$

(10), (11) and (12),

where $X$ and $Y$ are given matrices verifying the conditions of Theorem 1.

In order to prove this, it suffices to apply Schur's complement in (28) and, to pre and post multiply the obtained matrix inequality respectively by $\Theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ 0 & Y & V \end{bmatrix}$ and $\Theta'$.

It is worth noticing that matrix $V$ appears explicitly in (29). In this case, once $V$ is obtained, it should be verified *a posteriori* if it is indeed invertible. Alternatively, a constraint of type $V + V' > 0$ (or $< 0$) can be incorporated in the optimization problem to ensure that $V$ is nonsingular.

On the other hand, in practice, we are mainly interested in maximizing the region of stability in directions associated to the states of the plant. In this case the vectors $v_l$ assume the form $\begin{bmatrix} v'_{l1} & 0 \end{bmatrix}'$ and, from (28), it follows that $(i)$ in (29) can be replaced by the constraint:

$$v'_{l1}Yv_{l1} \leq \nu^{-1}\eta, \quad l = 1, \ldots, r \tag{30}$$

## 5.2 Performance Improvement

Let $\Xi$ be a given set in the state space, for which we want to ensure that $\forall \xi(0) \in \Xi$, $\xi(t) \to 0$ as $t \to \infty$. Among all the feasible controllers ensuring that, one may be interested in improving the performance of the closed-loop system.

A natural performance measure is given by the following quadratic criterion on plant states:

$$\mathcal{J}_Q = \int_0^\infty x(t)'Qx(t)dt \quad \text{where} \quad Q = Q' \geq 0, \quad Q \in \Re^{n \times n}.$$

In this case, if

$$\dot{V} + \frac{1}{\gamma}\xi' \begin{bmatrix} I \\ 0 \end{bmatrix} Q \begin{bmatrix} I & 0 \end{bmatrix} \xi < 0, \tag{31}$$

it follows that $\mathcal{J}_Q < \gamma V(0) < \gamma\nu^{-1}, \ \ \forall \xi\ (0) \in \mathcal{E}(P, \nu^{-1})$.

Actually, (31) is satisfied if:

$$\begin{bmatrix} \mathbf{J_1} & A+\hat{A}'+B\hat{D}C & X(Q^{1/2}) & \mathbf{J_2} & -BS+\hat{H}'_{\xi 1} \\ \star & He\{YA+\hat{B}C\} & (Q^{1/2}) & \mathbf{J_3} & \hat{E}+\hat{H}'_{\xi 2} \\ \star & \star & -\gamma I & 0 & 0 \\ \star & \star & \star & -2S_\Delta & \hat{H}'_\phi \\ \star & \star & \star & \star & -2S \end{bmatrix} < 0 \tag{32}$$

Another interesting performance criterion is the maximization of the exponential convergence of the trajectories. Note that if we ensure that

$$\dot{V} + \mu\xi' P\xi < 0 \tag{33}$$

it follows that $V(t) < e^{-\mu t}V(0), \ \ \forall \xi\ (0) \in \mathcal{E}(P, \nu^{-1})$. This fact guarantees exponential convergence of the trajectories to the origin with a rate given by $\mu$. The relation (33) is satisfied if

$$\begin{bmatrix} \mathbf{J_1}+\mu X & A+\hat{A}'+B\hat{D}C+\mu I & \mathbf{J_2} & -BS+\hat{H}'_{\xi 1} \\ \star & He\{YA+\hat{B}C\}+\mu Y & \mathbf{J_3} & \hat{E}+\hat{H}'_{\xi 2} \\ \star & \star & -2S_\Delta & \hat{H}'_\phi \\ \star & \star & \star & -2S \end{bmatrix} < 0, \tag{34}$$

Note that (34) ensures that all the eigenvalues of matrix $\mathbb{A}$ have real part smaller than $-\mu/2$.

The following convex optimization problem can therefore be formulated to take into account performance issues with a guaranteed region of stability:

$$\max \mu$$

subject to

$$\begin{bmatrix} \nu^{-1} & v'_{l1} & v'_{l1}Y + v'_{l2}V' \\ v_{l1} & X & I \\ Yv_{l1}+Vv_{l2} & I & Y \end{bmatrix} > 0 \tag{35}$$

$$l = 1, \ldots, r$$

(11), (12) and (32) (or (34))

When the objective is the maximization of the exponential convergence of the trajectories (i.e. (34) is considered), problem (35) can be efficiently solved as a GEVP ([3]).

**Remark 4** *Condition (34) can also be used to adapt the convex optimization problem (29) in order to maximize the size of $\mathcal{E}(P, \nu^{-1})$ while guaranteeing a pre-specified degree of exponential convergence inside it. For that, it suffices to replace condition (10) by (34), with a fixed value $\mu > 0$.*

### 5.3 Disturbance Tolerance

The idea consists in maximizing the bound on the disturbance, for which we can ensure that the system trajectories remain bounded. This can be accomplished by the following optimization problem.

$$\begin{array}{c} \min \delta \\ \text{subject to} \;\; (22), (23) \text{ and } (24) \end{array} \tag{36}$$

Note that, in this case, we are not interested in the value of $\gamma$. Indeed, $\gamma$ will assume a finite value to ensure that (22) is verified.

### 5.4 Disturbance Rejection

For an *a priori* given bound on the $\mathcal{L}_2$ norm of the admissible disturbances (given by $\frac{1}{\delta}$), the idea consists in minimizing the upper bound for the $\mathcal{L}_2$-gain of from $w(t)$ to $y(t)$. This can be obtained from the solution of the following optimization problem:

$$\begin{array}{c} \min \gamma \\ \text{subject to} \;\; (22), (23) \text{ and } (24) \text{ (or (26) and (27))} \end{array} \tag{37}$$

## 6 Numerical Examples

Consider the following data for the nonlinear system (1):

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -3 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \; D_w = 0$$
$$G = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; L = \begin{bmatrix} 1 & 0 \end{bmatrix}, \Omega = 1.4, \; u_0 = 2.$$

In the sequel we illustrate the application of the results of sections 3 and 4 as well as the optimization problems proposed in section 5.

### 6.1 Internal Stabilization Problem

Since $A$ is not Hurwitz, only local (regional) stabilization is possible. Thus, we first solve the convex problem (29), with $\mu = 1$ (see Remark 4), and with $\Xi = Co \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$[1]. This choice corresponds to maximize $\mathcal{E}(P, \nu^{-1})$ along the directions associated to the plant states.

[1]Considering that the eigenvalues of $\mathbb{A}$ are given by $a_i + jb_i$, an additional constraint to ensure that $\Re(a_i) \geq -10$ and $atan(\frac{|b_i|}{|a_i|}) \leq 45^o$ has been considered to guarantee well-conditioned solutions.

| $\nu$ | $z \in \tilde{\mathcal{S}}_1$, with $\rho = 1.8$ | | $z \in \Re^p$ | |
|---|---|---|---|---|
| | $\beta$ | Area | $\beta$ | Area |
| 10 | 1.7973 | 10.7317 | 4.7221 | 77.3880 |

Table 1: Enlargements for $\nu = 10$

For $\nu = 10$, Table 1 shows the obtained scaling factor of $\Xi$ and the area of the intersection of $\mathcal{E}(P, \nu^{-1})$ with the hyperplane defined by the plant states, denoted as $\mathcal{E}(Y, \nu^{-1})$ (the area is given by $\pi\sqrt{det((\nu Y)^{-1})}$)). Two cases are considered. In the first one the nonlinearity $\phi(\cdot)$ is suppose to be locally verified while in the second it is globally verified. As expected, a larger domain of stability is obtained when $\phi(\cdot)$ is globally verified. Figure 1 shows the cut in the plane defined by the plant states of ellipsoidal sets obtained for $\nu = 10$.

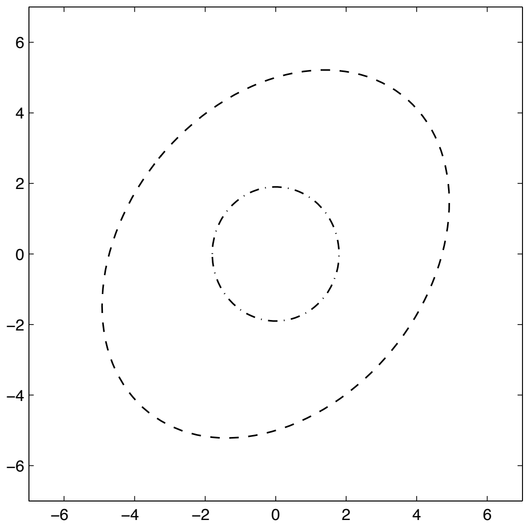


Figure 1: $\mathcal{E}(Y, 10^{-1})$ for $\phi$ locally (--) and globally (-.) verified

Let us now consider, for $\nu = 1$, the minimization of the upper-bound $\gamma$ for the quadratic criterion $\mathcal{J}_Q$, with $Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, by solving the convex optimization problem (35). In this case, we consider $\Xi = Co\left\{ \begin{bmatrix} \kappa \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \kappa \\ 0 \\ 0 \end{bmatrix} \right\}$, where $\kappa > 0$ may assume different values. Tables 2 and 3 show results obtained when $\phi(\cdot)$ is locally and globally verified, respectively[2]. By comparing the two tables, we observe again

[2] "-" denotes infeasibility of the LMIs

better results when the sector condition on $\phi(\cdot)$ is globally verified, since smaller guaranteed performances are obtained for greater values of $\kappa$ in Table 3. Now, in each table it can also be noticed a trade-off between the size of the guaranteed region of stability and the upper-bound for $\mathcal{J}_Q$. In fact, the greater is $\kappa$ and, in consequence, the area of the obtained $\mathcal{E}(Y, \nu^{-1})$, the worse is the upper-bound obtained for $\mathcal{J}_Q$.

| $\kappa$ | $\gamma$ | Area |
|---|---|---|
| 1 | 0.1788 | 3.3449 |
| 1.5 | 0.7395 | 7.5471 |
| 1.79 | 1.4406 | 10.2270 |
| 1.80 | - | - |

Table 2: Guaranteed quadratic performance with $\nu = 1$ and for $\phi$ locally verified

| $\kappa$ | $\gamma$ | Area |
|---|---|---|
| 1 | 0.1739 | 3.3698 |
| 3 | 11.1319 | 30.8500 |
| 4.73 | 42958 | 77.6659 |
| 4.74 | - | - |

Table 3: Guaranteed quadratic performance with $\nu = 1$ and for $\phi$ globally verified

Table 4 shows, for $\kappa = 1$ and $\nu = 1$, the controllers parameters when $\phi(\cdot)$ is locally or globally verified.

| | $z \in \tilde{\mathcal{S}}_1$, with $\rho = 1.8$ | $z \in \Re^p$ |
|---|---|---|
| $A_c$ | $\begin{bmatrix} -7.9032 & 0.000845 \\ 0.99099 & -2.3811 \end{bmatrix}$ | $\begin{bmatrix} -7.8989 & 0.002745 \\ 0.61055 & -2.382 \end{bmatrix}$ |
| $B_c$ | $\begin{bmatrix} 0.17907 \\ 349.6523 \end{bmatrix}$ | $\begin{bmatrix} 0.3777 \\ 346.2083 \end{bmatrix}$ |
| $C_c$ | $\begin{bmatrix} -0.0041914 & -0.0073054 \end{bmatrix}$ | $\begin{bmatrix} -0.0011032 & -0.0074306 \end{bmatrix}$ |
| $D_c$ | -4.8779 | -4.8859 |
| $E_c$ | $\begin{bmatrix} -0.22739 \\ -92.0092 \end{bmatrix}$ | $\begin{bmatrix} -0.37372 \\ -90.2543 \end{bmatrix}$ |
| $F_c$ | -0.10961 | -0.2020 |
| $G_c$ | $\begin{bmatrix} -0.12097 \\ -80.0964 \end{bmatrix}$ | $\begin{bmatrix} 1.5226 \\ -72.0697 \end{bmatrix}$ |

Table 4: Controllers parameters for $\kappa = 1$

## 6.2 Disturbance Tolerance and Rejection

Considering the optimization problem (36), the minimal value of $\delta$ (i.e. the maximal bound on the input disturbance) for which it is possible to find a controller (considering the conditions given by Theorem 2) that ensures the state trajectories are bounded, is given by 0.4471 .

On the other hand, considering that the bound on the admissible disturbances is given by $\delta^{-1}$, in Table 5 the obtained values for $\gamma$ solving problem (37) for different values of $\delta$ are shown. Note that, as expected, greater is $\delta$ (smaller is the admissible given disturbance bound), smaller is the upper bound on the $\mathcal{L}_2$ gain from $w$ to $y$ (i.e. higher is the disturbance rejection).

| $\delta$ | $\gamma$ |
|---|---|
| 0.4471 | 2695.2 |
| 0.4500 | 87.6956 |
| 0.5000 | 6.1169 |
| 0.6000 | 1.9992 |
| 0.7000 | 1.0088 |
| 0.8000 | 0.6119 |
| 0.9000 | 0.4122 |
| 1.0000 | 0.2969 |
| 2.0000 | 0.0484 |
| 10.0000 | 0.0015 |

Table 5: Tradeoff $\gamma \times \delta$

For $\delta = 1$, from the solution of problem (37), the following controller matrices are obtained[3].

$$A_c = \begin{bmatrix} -11.4627 & -4.4814 \\ -141.2563 & -73.1282 \end{bmatrix}, \; B_c = \begin{bmatrix} -21780.8804 \\ 43561.7786 \end{bmatrix}, \; E_c = \begin{bmatrix} -0.0051229 \\ -0.0068569 \end{bmatrix},$$

$$Gc = \begin{bmatrix} 29041.3842 \\ -58082.7748 \end{bmatrix}, \; C_c = \begin{bmatrix} -1254.7592 & -627.3797 \end{bmatrix},$$

$$D_c = \begin{bmatrix} -47.266 \end{bmatrix}, \; Fc = \begin{bmatrix} -0.027384 \end{bmatrix}$$

In order to illustrate the dynamic behavior achieved with this controller, we consider that the nonlinearity is given by

$$\phi(z) = \frac{1.4 \times 1.8}{\pi} \sin\left(\frac{\pi}{1.8} z\right)$$

[3]Considering that the eigenvalues of $\mathbb{A}$ are given by $a_i + jb_i$, an additional constraint to ensure that $\Re(a_i) \geq -100$ and $atan(\frac{|b_i|}{|a_i|}) \leq 45^o$ has been considered to guarantee well-conditioned solutions.

Note that, considering that $z \in \tilde{\mathcal{S}}_1$ with $\rho = 1.8$, this nonlinearity belongs locally to a sector defined by $\Omega = 1.4$.

Figures 2 and 3 depict the simulation results for the closed-loop system, considering $w(t)$ as a pulse signal defined as follows:

$$w(t) = \begin{cases} 10 & \text{if} \quad 0 < t \leq 0.01 \\ 0 & \text{if} \quad t > 0.01 \end{cases}$$

which gives $||w||_2^2 = 1$. Although the control signal remains saturated for some time after disturbance action stops, it can be seen that the disturbance is rejected. This shows that the controller ensures the trajectories remain inside the region of attraction of the closed-loop system

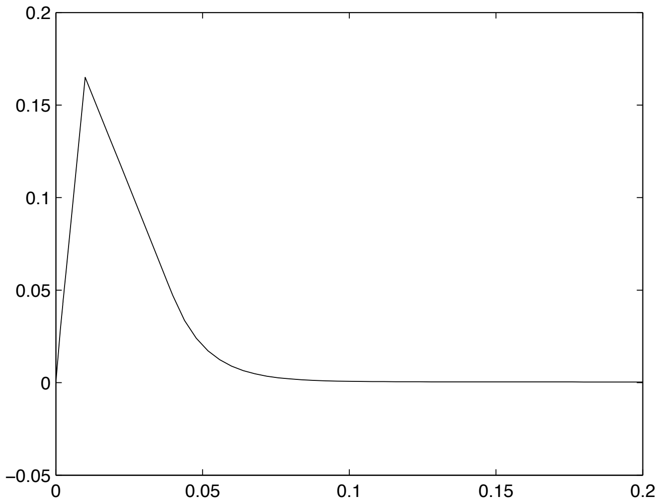


Figure 2: System Output

# 7 Conclusion

In the present work we have addressed the stabilization problem of Lur'e type nonlinear systems subject to input saturation. Constructive LMI results allowing to compute a nonlinear dynamic controller having as inputs both the plant output and the output of the dynamic nonlinearity have been proposed. The results regards the internal (asymptotic) as well as the external (input-to-state and input-to-output) stabilization of the closed-loop system, both in regional and global contexts.

Considering the internal stabilization, convex optimization problems with LMI constraints to compute the controller aiming at maximizing an ellipsoidal estimate of the closed-loop domain of attraction or at improving the performance of the closed-loop system while guaranteeing a pre-specified region of stability have been formulated. Regarding the external stabilization, the LMI conditions are cast in optimization problems to compute a controller to maximize the bound on the $\mathcal{L}_2$ norm of the admissible disturbances or,

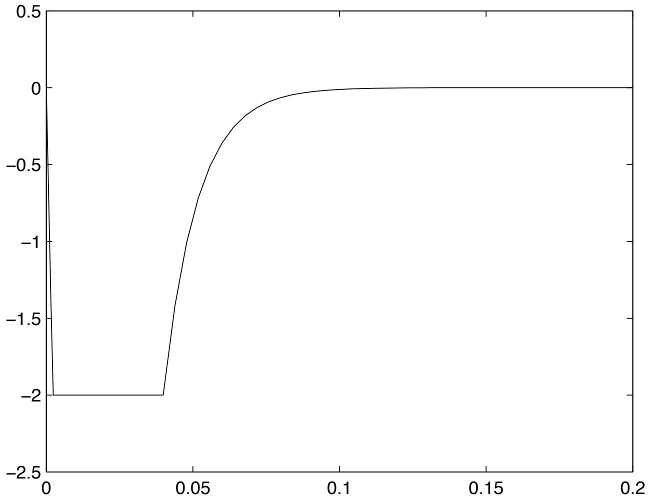


Figure 3: Controller Output

when this bound is given, to design the controller in order to minimize the $\mathcal{L}_2$-gain between the disturbance and the regulated output.

Thus, the proposed approach provides a systematic way to compute dynamic stabilizing controllers, along with a formal characterization of sets of admissible initial states and disturbances, for the class of Lur'e type nonlinear systems in the presence of saturating actuators.

# References

[1] H. K. Khalil. *Nonlinear Systems - Third Edition.* Prentice Hall, 2002.

[2] M. R. Liberzon. Essays on the absolute stability theory. *Automation and Remote Control*, 67(10):1610–1644, 2006.

[3] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear Matrix Inequalities in System and Control Theory.* SIAM Studies in Applied Mathematics, 1994.

[4] C. Pittet, S. Tarbouriech, and C. Burgat. Stability regions for linear systems with saturating controls via circle and popov criteria. In *36th IEEE Conf. on Decision and Control (CDC'97)*, pages 4518–4523, San Diego, USA, 1997.

[5] H. Hindi and S. Boyd. Analysis of linear systems with saturation using convex optimization. In *37th IEEE Conf. on Decision and Control (CDC'98)*, pages 903–908, Tampa, USA, 1998.

[6] T. Hu, Z. Lin, and B. M. Chen. An analysis and design method for linear systems subject to actuator saturation and disturbance. *Automatica*, 38:351–359, 2002.

[7] V. Kapila and W.M. Haddad. Fixed-structure controller design for systems with actuator amplitude and rate nonlinearities. *Int. Journal of Control*, 73(6):520–530, 2000.

[8] T. Kiyama and T. Iwasaki. On the use of multi-loop circle for saturating control synthesis. *Systems & Control Letters*, 41:105–114, 2000.

[9] E. F. Mulder, M. V. Kothare, and M. Morari. Multivariable anti-windup controller synthesis using linear matrix inequalities. *Automatica*, pages 1407–1416, Sep. 2001.

[10] G. Grimm, J. Hatfield, I. Postlethwaite, A. Teel, M. Turner, and L. Zaccarian. Antiwindup for stable systems with input saturation: an lmi-based synthesis. *IEEE Trans. on Automatic Control*, 48(9):1500–1525, 2003.

[11] J. M. Gomes da Silva Jr. and S. Tarbouriech. Anti-windup design with guaranteed regions of stability: an lmi-based approach. *IEEE Trans. on Automatic Control*, 50(1):106–111, 2005.

[12] E. B. Castelan, S. Tarbouriech, and Isabelle Queinnec. Stability and stabilization of a class of nonlinear systems with saturating actuators. In *Proc. of 16th IFAC World Congress in Automatic Control*, Prague, 2005.

[13] E. B. Castelan, U. Moreno, and E. R. de Pieri. Absolute stabilization of discrete-time systems with a sector bounded nonlinearity under control saturations. In *IEEE International Symposium on Circuits and Systems (ISCAS 2006)*, pages 3105–3108, Greece, 2006.

[14] E. B. Castelan, S. Tarbouriech, and Isabelle Queinnec. Control design for a class of nonlinear continuous-time systems. *Automatica*, 2008 (to appear).

[15] E. B. Castelan, J. Corso, and U. F. Moreno. Stability and stabilization of a class of uncertain nonlinear discrete-time systems with saturating actuators. In *Symposium on System, Structure and Control, SSSC2007*, Foz do Iguaçu - PR, Brazil, Outubro 2007.

[16] M. Arcak, M. Larsen, and P. Kokotovic. Circle and popov criteria as tools for nonlinear feedback designs. *Automatica*, 39:643–650, 2003.

[17] M. Arcak and P. Kokotovic. Feasibility conditions for circle criterion designs. *Systems & Control Letters*, 42:405–412, 2001.

[18] T. Kiyama, S. Hara, and T Iwasaki. Effectiveness and limitation of circle criterion for lti robust control systems with control input nonlinearities of sector type. *International Journal of Robust and Nonlinear Control*, (15):873–901, 2005.

[19] C. Scherer, P. Gahinet, and M. Chilali. Multiobjective output-feedback control via lmi optimization. *IEEE Trans. on Automatic Control*, 42(7):896–911, 1997.