Tuning Rules for Proportional Resonant Controllers

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***Abstract*—In this paper we propose a particular structure for resonant controllers and a tuning method of the Ziegler-Nichols type for their tuning. Performance criteria for resonant controllers are also defined. The effectiveness of the tuning rules is illustrated by their application and corresponding performance assessment in a test batch consisting of four representative classes of processes. The control performance is analyzed in detail for one particular example, shedding light on the virtues and limitations of the control structure and of the tuning method.**

I. Introduction

The Internal Model Principle is a basic concept in control systems that can be found in most control textbooks. It states that in order to track a given reference, or to reject a given disturbance, the controller must contain the reference or disturbance model. When the references or disturbances are steps, which is by far the most common case, this internal model is formed by an integrator, hence the widespread use of PID controllers. Another class of references and disturbances commonly found are those of sinusoidal nature. In this case the controller must have a resonance frequency equal to that of the reference or disturbance, hence the denomination of resonant controller. Some applications of resonant controllers include Uninterruptible AC Power Sources (UPSs) [9], [7], active filters [3], engine fuel injection [8], and vibration control in flexible structures [4], [6]. In this last class of applications the controller reduces the resonant peaks of the loop's frequency response through the application of high-Q resonant controllers. Typically, the controller parameters are tuned considering the knowledge of the process model, unlike PID controllers, whose tuning can be made with a set of experimental rules.

Although the internal model principle is a well known fact taught at basic undergraduate courses in control systems, it does not necessarily represent the standard industrial practice. It is not rare to find practical applications in which PID controllers are favored even when the internal model principle prescribes the application of a resonant controller. Sometimes this choice is simply unjustified, and steady-state tracking errors result. In other cases the use of PID controllers in lieu of resonant controllers is justified by reformulating the control problem such that the reference and/or disturbance becomes a step. It is the case, for instance, in many applications involving UPS's (uninterruptible AC power sources): instead of following the sinusoidal reference signal, the phase and amplitude of this sinusoid are calculated and control loops for each one of these variables (which are of the step type) are implemented with PID controllers. This approach not only complicates the implementation but also causes delays in the control loop due to the computation of RMS value, deteriorating performance and reducing stability margins. Only recently resonant controllers for UPSs started appearing in the literature [9], but to this date this does not seem to have become the dominant industrial practice.

One of the reasons for PID controllers to be so popular is the existence of easily understandable and easily computable tuning rules, that can be implemented even by the layman or made algorithmic for application in auto-tuning. The celebrated Ziegler-Nichols tuning formulae have been proposed more than 70 years ago and are still widely applied [10]. On the other hand there are standard structures for PI(D) controllers and simple performance measures for step references and disturbances that are well established since a long time. In this paper we propose a standard parametrized structure called Proportional Ressonant - PR controller, with standard performances measures and, mainly, tuning rules that generalize those commonly used for PI controllers. The proposed method can be implemented experimentally in a straightforward manner, through the same standard relay feedback experiment that is commonly applied in the PI and PID cases, thus inheriting the simplicity that is probably the main virtue of the Ziegler-Nichols and related methods.

A batch of representative processes has been proposed by Astrom and Hagglund in [1] for assessing tuning rules for PID controllers. We assess our tuning formulae for PR controllers using four different classes of processes from this test batch, with varying parameter values, resulting in 10 different processes. Counting the different reference frequencies and different tuning points tested for each process, we have performed 76 distinct and representative tests. These are presented in Section V, where we analyze in detail one class of processes, applying two different tuning rules for five different reference frequencies, and in Section VI where we summarize the performance results for the other three classes of processes.

II. Preliminaries

A. Notation

The lower case letters $u(t)$, $y(t)$, $e(t)$, $r(t)$, $q_u(t)$ and $q_y(t)$ indicate the time domain signals in the control loop, whereas capital letters indicate their corresponding Laplace transforms - respectively $U(s)$, $Y(s)$, $E(s)$, $R(s)$, $Q_u(s)$ and $Q_y(s)$. We consider linear time invariant causal (LTIC) plants which are described in the Laplace domain by

$$Y(s) = G(s)[U(s) + Q_u(s)] + Q_y(s) \tag{1}$$

where $G(s)$ is the plant's transfer function, $U(s)$ is the control input, $Y(s)$ is the plant's output (the controlled variable),

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$Q_u(s)$ and $Q_y(s)$ are the disturbances at the plant's input and output respectively. The plant is controlled with unitary feedback by an LTIC controller, that is:

$$E(s) = R(s) - Y(s) \quad (2)$$
$$U(s) = C(s)E(s) \quad (3)$$

where $R(s)$ is the reference, $E(s)$ is the tracking error and $C(s)$ is the controller's transfer function.

The control objective is to track a given class of reference signals with zero tracking error in steady-state for a given class of disturbances. In other words, the closed-loop system (1)-(2)-(3) must be stable and satisfy $\lim_{t\to\infty} e(t) = 0$.

B. Time domain performance assessment

Performance of control systems is usually specified by criteria defined in terms of the system's response to a step reference signal: the settling time t_s and the maximum overshoot Mo.

1) Constant reference: Let the reference be a step, that is

$$r(t) = 0 \; \forall t < 0 \quad r(t) = r \; \forall t > 0,$$

where r is some given constant. The settling time is defined as the smallest time at which the tracking error becomes sufficiently small, that is:

$$t_s = \min_{t_1} : \quad | \, e_n(t) \, | < \epsilon \quad \forall t > t_1 \quad (4)$$

where $e_n(t) \triangleq \frac{e(t)}{r}$ is the normalized tracking error and ϵ is a user defined tolerance - usually $\epsilon \in [0.02; 0.05]$. The maximum overshoot Mo is defined as follows. Let M be the maximum amplitude achieved by the output during its transient response to a step reference:

$$M = \max_t \, | \, y(t) \, |$$

then

$$Mo = \max(\frac{M - | \, r \, |}{| \, r \, |}, 0)$$

2) Sinusoidal reference: Let us now define similar measures for sinusoidal reference signals, in order to assess the performance of the corresponding controllers in a systematic and standard-compliant way. Let $a_r > 0$ be the amplitude of the sinusoidal reference, that is,

$$r(t) = 0 \; \forall t < 0 \quad r(t) = a_r \sin(\omega_r t) \; \forall t > 0$$

for some frequency $\omega_r \in \mathbb{R}$. Replace r by a_r and then define settling time and overshoot in the same way as for the step reference, as follows. For the settling time, use definition (4) with $e_n(t) = \frac{e(t)}{a_r}$. It may be convenient to express the settling time in number of periods of the reference instead of (or in addition to) time units, that is, to use the following measure:

$$n_s = \frac{t_s \omega_r}{2\pi} \quad (5)$$

For the overshoot, redefine Mo as

$$Mo = \max(\frac{M - a_r}{a_r}, 0). \quad (6)$$

III. The Proportional Resonant Controller

In this Section the structure of the PR controller is presented, inspired and justified by the classical proportional integral (PI) control.

A. The PI controller

A PI controller generates two control actions then combines them to form the signal to be applied to the plant:

$$U(s) = U_i(s) + U_p(s). \quad (7)$$

In (7) $U_i(s)$ is the integral control action and $U_p(s)$ is the proportional action. These are defined as

$$U_i(s) = \frac{k_i}{s} E(s)$$
$$U_p(s) = k_p E(s)$$

Hence the transfer function of a PI controller is given by

$$C_{pi}(s) \triangleq \frac{U(s)}{E(s)} = \frac{k_i}{s} + k_p \quad (8)$$
$$= k_i C_i(s) + k_p C_p(s) \quad (9)$$

with seemingly obvious definitions for $C_i(s)$ and $C_p(s)$.

The fundamental role of a PI controller is to guarantee zero steady-state error when the reference and the disturbances acting on the plant are piecewise constant. This is achieved by the integral action alone, and the proportional action is zero in steady state. Hence, the most fundamental control action is the integral action and it could, at least in principle, be applied by itself to achieve this goal for a large class of processes. Indeed, any stable minimum phase process can be controlled by a purely integral controller, provided that sufficiently small integral gain is used. However, transient performance and stability margins with purely integral control could, and usually would, be unacceptably poor. Hence an additional term is added to the control law in order to improve transient performance and robustness, and to achieve closed-loop stability for a larger class of processes. This additional control action is the proportional action, which can be seen as the derivative of the most fundamental integral action, that is:

$$C_p(s) = sC_i(s) \quad (10)$$

B. The Proportional Resonant (PR) Structure

Consider now a different class of references to be tracked and disturbances to be rejected: sinusoids with a fixed frequency ω_r. The fundamental task of the resonant controller is performed by the following, fundamental, control action:

$$C_r(s) = \frac{s}{s^2 + \omega_r^2} \quad (11)$$

where ω_r is the frequency that must be followed and/or rejected. The integral control action $C_i(s)$ is a particular case of the resonant control action (11), obtained by setting $\omega_r = 0$. In vibration control, the fundamental control action is slightly more general, containing the model of a damped sinusoid instead of a pure one as in (11) [4], [6]. Like the integral action in the constant reference case, this fundamental control action is capable by itself of providing the basic stability plus zero steady-state tracking error for a large class of processes.

Like in the constant reference case, additional control actions must be included in order to achieve appropriate performance. To define such an additional control action, let us apply the same principle as in PI controllers, expressed in (10). Taking the derivative of the fundamental (resonant) control action yields:

$$C_p(s) = sC_r(s) \tag{12}$$

which added to the resonant control action results in the Proportional Resonant (PR) controller:

$$C_{pr}(s) = k_r \frac{s}{s^2 + \omega_r^2} + k_p \frac{s^2}{s^2 + \omega_r^2}. \tag{13}$$

where $k_r, k_p \in \mathbb{R}$ are parameters to be tuned.

IV. Ziegler-Nichols tuning

A. Classical tuning rules for PID controllers

One of the reasons for PID controllers to be so popular is the existence of easily understandable and easily computable tuning rules, that can be applied even by the layman or made algorithmic for application in auto-tuning. Huge amounts of literature have been produced on PID tuning rules and a myriad of methods have been proposed and successfully applied, such as λ-tuning, Cohen-Coon, MIGO, etc, to name just a few - see [1] for a thorough overview. These methods, even those proposed for multivariable processes [2], constitute variations of the methods proposed in the seminal work [10]. In [10], J.G. Ziegler and N.B. Nichols proposed a tuning method that consists in causing an oscillation in closed-loop, measuring the oscillations' frequency and amplitude and then applying simple formulas for each controller parameter, formulas that involve these measurements. In this paper this method, to be called the *forced oscillation method*, will be explored and adapted to PR controllers.

The forced oscillation method is based solely on the knowledge of the *ultimate point* of the plant's frequency response. The ultimate point for a given transfer function is the point at which its Nyquist plot crosses the negative real axis - equivalently, the point corresponding to the smallest frequency for which its phase reaches the value $-\pi$. The characteristics of the ultimate point are the ultimate frequency ω_u and the ultimate gain K_u, which are defined as

$$\begin{aligned} \omega_u &= \min_{\omega \geq 0} \omega : \angle G(\jmath\omega) = -\pi \\ K_u &= \frac{1}{\mid G(\jmath\omega_u) \mid}. \end{aligned}$$

With these definitions, the most contemporary interpretation of the forced oscillation method can be summarized as follows.

1) identify the ultimate point of the process' frequency response, that is, determine ω_u and K_u;
2) choose the parameters of the controller such that $C(\jmath\omega_u)G(\jmath\omega_u) = p$, or equivalently
$$C(\jmath\omega_u) = -K_u p, \tag{14}$$
where p is a prespecified location in the complex plane.

The first step of the method is usually performed by means of a relay feedback experiment. Relay feedback consists in a closed-loop experiment in which the following nonlinear control is applied:

$$u(t) = d\ sign(e(t)) + b. \tag{15}$$

In (15) $sign(\cdot)$ is the sign function ($sign(x) = 1$ for positive x and $sign(x) = -1$ for negative x), $d \in \Re^+$ is a parameter to be chosen and $b \in \Re$ is the bias. The bias parameter b must be adjusted so that the oscillation is symmetric. In most practical applications a hysteresis is added to the relay feedback (15) in order to prevent spurious switching due to noise. Once a symmetric oscillation is obtained in the relay experiment, its amplitude A_u and period T_u are measured and the ultimate quantities are calculated from [1]

$$K_u = \frac{4d}{\pi A_u} \quad \omega_u = \frac{2\pi}{T_u} \tag{16}$$

The second step of the method is accomplished by solving equation (14) for the controller's gains k_i, k_p, k_d with the chosen location p. This is a complex equation with two unknowns for PI and three unknowns for PID. Thus for PID controllers there is one degree of freedom in the choice of the controller's parameters, which is usually removed by imposing the additional constraint that the two zeros of the controller's transfer function are the same.

Under the reasonable assumption that the frequency response of the process is sufficiently smooth, shifting the ultimate point away from -1 in the complex plane implies shifting the whole frequency response away from it, thus leading to good stability margins. Different locations p have been proposed over the years, each one providing different transient performance and stability margins. The original Ziegler-Nichols tuning formulas in [10] correspond to

$$p = -0.4 + \jmath 0.08 \tag{17}$$

for PI controllers. Another famous set of formulae is the one by Tyreus and Luyben, which corresponds to shifting the ultimate point to

$$p = -0.31 + \jmath 0.023 \tag{18}$$

for PI controllers, usually resulting in more conservative tuning than the Ziegler-Nichols formulae [5].

B. Tuning of PR controller

The forced oscillation method is very practical for experimental implementation, which is an important reason behind its widespread application. In order to obtain an equally practical method for the tuning of PR controllers, let us apply the rationale of the forced oscillation method to this class of controllers. The frequency response of the PR controller is

$$C(\jmath\omega) = \frac{\jmath k_r \omega - k_p \omega^2}{\omega_r^2 - \omega^2} \tag{19}$$

Substituting (19) into the tuning equation (14) yields

$$\frac{\jmath k_r \omega_u - k_p \omega_u^2}{\omega_r^2 - \omega_u^2} = -K_u Re(p) - \jmath K_u Im(p) \tag{20}$$

or, isolating the controller's gains:

$$k_r = K_u Im(p) \frac{\omega_u^2 - \omega_r^2}{\omega_u} \quad k_p = K_u Re(p) \frac{\omega_r^2 - \omega_u^2}{\omega_u^2} \quad (21)$$

where $Re(p)$, $Im(p)$ are the real and imaginary parts of p.

It is immediately seen that for $\omega_r \approx \omega_u$ the controller's gains will be close to zero, and poor performance is to be expected in this case. It is important to notice that this is not a limitation of the tuning rules just proposed; it is rather a limitation of the controller structure. Indeed, when $\omega_r \approx \omega_u$ the controller structure presents very large gains in a range around the ultimate frequency of the process, which is only slightly changed by it. Thus the loop transfer function will necessary have very large gains around the negative real axis. This will become clearer in the frequency response analysis of the example to be presented in Section V.

It is also worth noticing that $Re(p) < 0$, thus for $\omega_r > \omega_u$ the proportional gain will be negative. This seems rather odd from a practical point of view, but so is the attempt to track a reference of such a high frequency compared to the bandwidth of the process. Nevertheless the gains provided by (21) yield satisfactory performance, even when negative, in the many examples to be presented in Sections V and VI.

In order to validate the proposed methodology for tuning the parameters of PR controllers, it will be applied to a batch of representative processes, extracted from [1], in Section VI, using both the Ziegler-Nichols point (p defined in (17)) and the Tyreus-Luyben point (p defined in (18)). But first, we detail its application to one particular process in Section V.

V. A BENCHMARK CLASS OF PROCESSES

In order to illustrate the features of the proposed tuning methodology, a specific process class was chosen from the set of processes presented by Astrom and Hagglund in [1]:

$$G(s) = \frac{e^{-s}}{(sT+1)^2}. \quad (22)$$

In this Section the application to this class of processes of the tuning rules proposed is explored in detail.

A. A particular case

In this Subsection detailed results are presented and discussed for the particular process (22) with $T = 10$ *sec*. Five different reference sinusoidal signals, with unit amplitude and with different frequencies, have been considered.

The first step for application of the tuning method is to define the amplitude and the frequency of the output process variable considering the relay experiment. For this case, the parameters of the equation (15) were defined as $b = 1$, which will result in a symmetric oscillation for a unit step reference, and $d = 2$. The signal presented in Figure 1 represents the process' output signal considering as reference input a step with amplitude one. Based on this experiment it is possible to determine the amplitude and the period of the self oscillating part of the signal as $A_u = 0.13$ and $T_u = 15$ seconds. This information is used to calculate the ultimate gain K_u

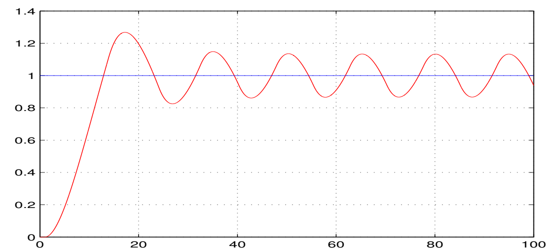
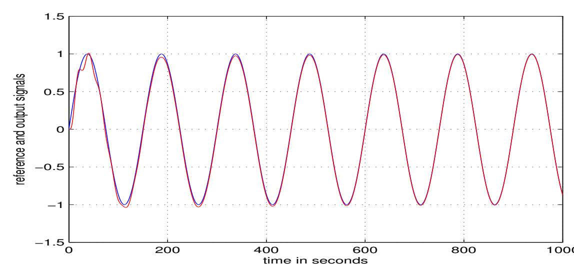


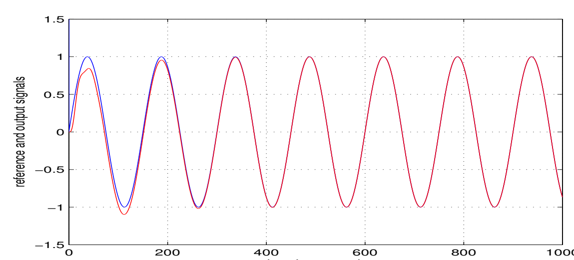
Fig. 1. Closed-loop response for the relay experiment.

and ultimate frequency ω_u from equations (16), resulting in $K_u = 19.58$ and $\omega_u = 0.42 rad/s$.

For each different frequency of the reference signal the PR tuning is different, according to the tuning formulae (21). Consider initially a reference with frequency $\omega_r = 0.1\omega_u$; then the controller gains are calculated from (21) as $k_p = 7.75$ and $k_r = 0.65$ for the ZN point, and $k_p = 6.01$ and $k_r = 0.18$ for the TL point. The resulting reference and the output signals for each set of gains are presented, respectively, in Figures 2(a) and 2(b).



(a) ZN point



(b) TL point

Fig. 2. Reference and output signals for $\omega_r = 0.1\omega_u$

A frequency response analysis provides a clearer picture. The Nyquist diagrams of the process' transfer function and of the loop transfer function with each one of the controllers designed are shown in Figures 3 and 4. Only the loop transfer function with Ziegler-Nichols (ZN) tuning in shown in Figure 3 since in this scale the loop transfer function with Tyreus-Luyben (TL) tuning would look the same and the process' transfer function would not be discernible because of its finite size. Notice that because of the controller's poles at the imaginary axis the Nyquist diagram tends to infinity at these frequencies $\pm\omega_r$ (hence the absence of scale in this plot). A zoom of this plot is shown in Figure 4, where it can be seen that the Nyquist diagrams of the loop transfer functions with the two controllers do not encircle the point -1 and that the Tyreus-Luyben tuning provides larger stability margins, as expected. The frequency response is smooth enough around the negative real axis so that shifting the ultimate point away from -1 guarantees good stability margins.

Consider now a frequency closer to the ultimate frequency: $\omega_r = 0.5\omega_u$. The corresponding Nyquist diagrams are shown in Figure 5; only the zoomed version is presented in this case. It is seen that the proximity of the reference's frequency to the ultimate frequency causes the diagram to pass close to -1. Here, shifting the ultimate point away from -1 does not guarantee good stability margins; the nearby points are not shifted along because the frequency response is not sufficiently smooth in this range of frequencies. The stability margins are much smaller than in the previous case ($\omega_r = 0.1\omega_u$) and, accordingly, poorer transient response is to be expected. Indeed, this is confirmed by the time domain results presented in Figures 6(a) and 6(b), where the system's response to a reference with frequency $\omega_r = 0.5\omega_u$ is presented for each controller setting (ZN and TL); the corresponding performance measures are presented in Tables I and II.

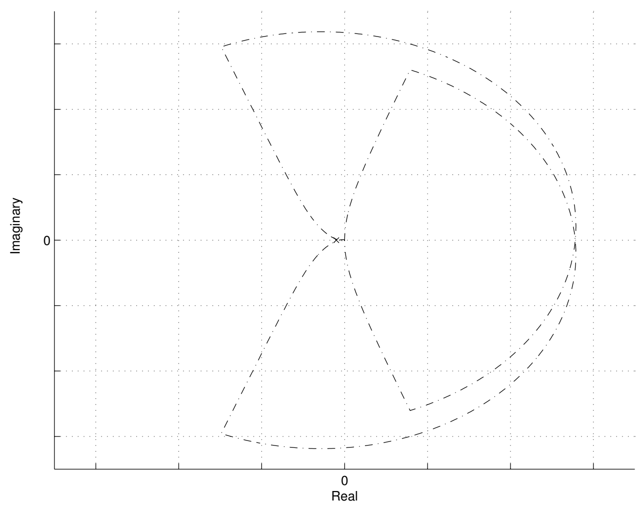


Fig. 3. Nyquist diagram of the loop transfer function for $\omega_r = 0.1\omega_u$ with ZN tuning; the semicircles in the graph have infinite radius

Three additional experiments were performed with $\omega_r = 0.2\omega_u$, $\omega_r = 2\omega_u$ and $\omega_r = 5\omega_u$. The simulation results are presented in Figures 7(a) to 9(b), while the controller's settings and the resulting performance measures (settling times and overshoots) appear in Tables I (for ZN point) and II (for TL point). The controller's gains and performance measures in these Tables summarize the overall performance of the closed loop system for a wide range of frequencies of the reference signal. As expected, the performance deteriorates as the frequency of the reference approaches the ultimate frequency. For frequencies within one octave of the ultimate frequency the performance is poor, even becoming unstable in some cases (not shown in these Tables). It is worth noting that for frequencies above the ultimate frequency the gains are negative - which is clear in the formulae (21).

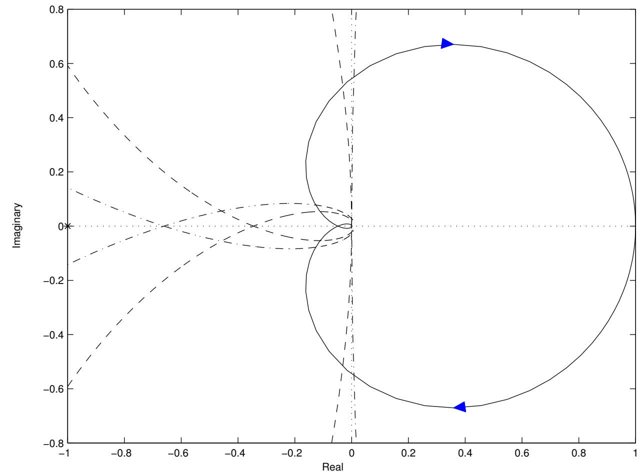


Fig. 4. Nyquist diagrams for $\omega_r = 0.1\omega_u$: process (full line), TL tuning (dashed) and ZN tuning (dash-dot)

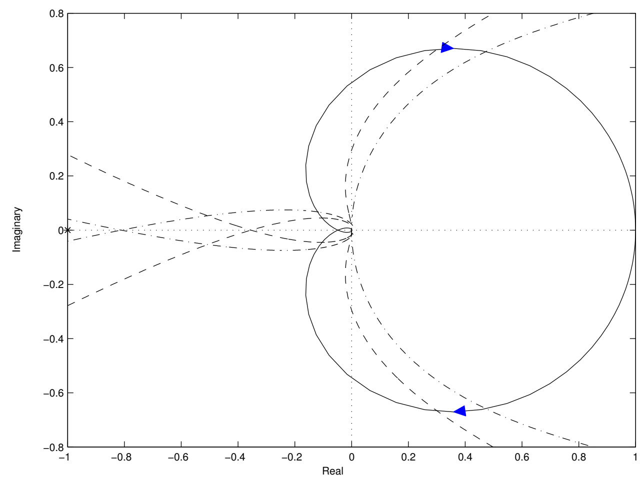


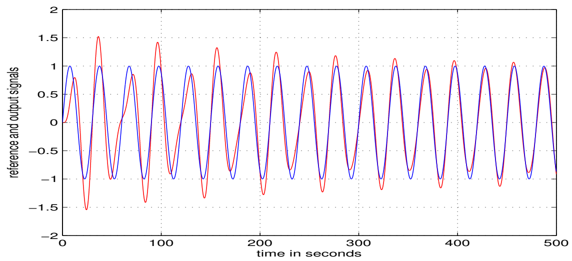
Fig. 5. Nyquist diagrams for $\omega_r = 0.5\omega_u$: process (full line), TL tuning (dashed) and ZN tuning (dash-dot)

TABLE I
PERFORMANCE MEASUREMENTS - ZN POINT

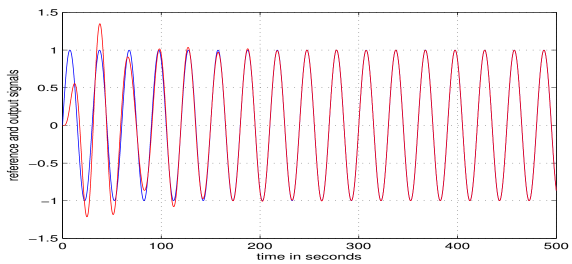
ω_r	k_p	k_r	n_s	M_o
0.1 ω_u	7.75	0.65	2.83	0.00
0.2 ω_u	7.52	0.63	2.81	0.14
0.5 ω_u	5.87	0.49	32.77	0.54
2.0 ω_u	-23.50	-1.96	5.78	0.21
5.0 ω_u	-188.04	-15.75	3.63	0.13

B. Different time constants

Another set of tests were performed with the process class (22) using three other different values for the parameter T. Tables III and IV summarize the results obtained for the process parameter $T = 0.5$ seconds, whose ultimate frequency is $\omega_u = 1.74$ rad/s. Tables V and VI summarize the results for $T = 1.0$ seconds, with $\omega_u = 1.34$ rad/s. Finally, Tables VII and VIII summarize the results for $T = 5.0$, with $\omega_u = 0.59$

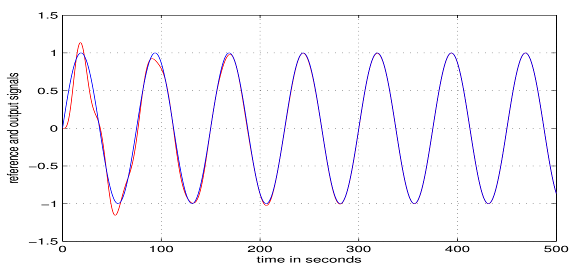


(a) ZN point

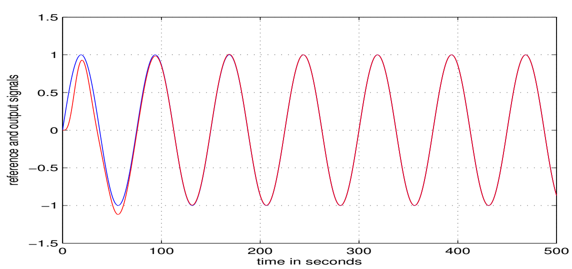


(b) TL point

Fig. 6. Reference and output signals for $\omega_r = 0.5\omega_u$.



(a) ZN point

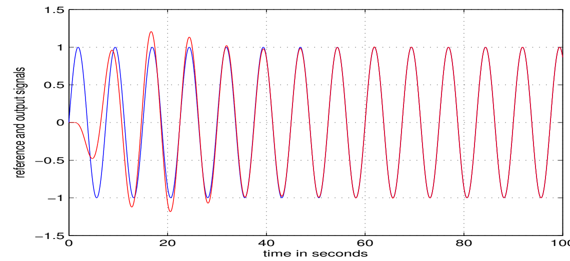


(b) TL point

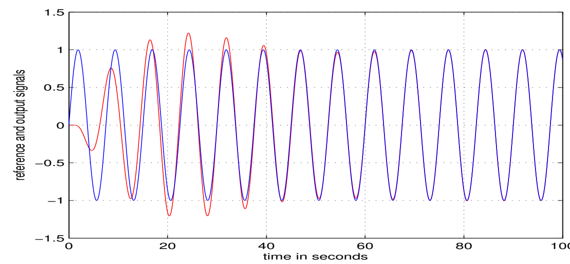
Fig. 7. Reference and output signals for $\omega_r = 0.2\omega_u$.

TABLE II
PERFORMANCE MEASUREMENTS - TL POINT

ω_r	k_p	k_r	n_s	M_o
0.1 ω_u	6.01	0.18	1.71	0.09
0.2 ω_u	5.82	0.18	1.22	0.12
0.5 ω_u	4.55	0.14	5.63	0.35
2.0 ω_u	-18.21	-0.56	7.86	0.22
5.0 ω_u	-145.73	-4.53	4.30	0.06

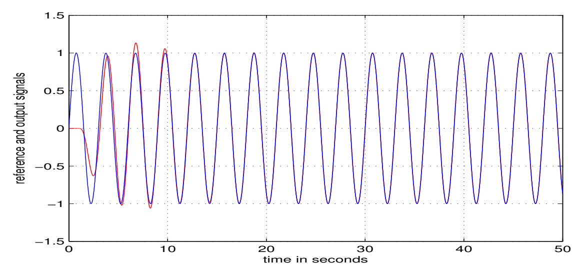


(a) ZN point

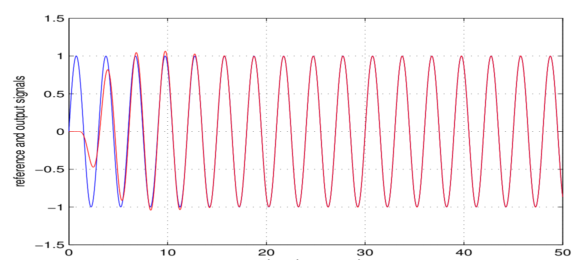


(b) TL point

Fig. 8. Reference and output signals for $\omega_r = 2\omega_u$.



(a) ZN point



(b) TL point

Fig. 9. Reference and output signals for $\omega_r = 5\omega_u$.

rad/s. The blank cells appearing in some of these tables denote an unstable closed loop system. It is observed that appropriate performance is obtained in most cases.

VI. A TEST BATCH

The tuning formulas have also been tested for the following three classes of processes from the test batch proposed by Astrom and Hagglund in [1]:

$$G_1(s) = \frac{1}{(s+1)((sT)^2 + 1.4Ts + 1)} \tag{23}$$

TABLE III
PERFORMANCE MEASUREMENTS ($T = 0.5$ S) - ZN POINT

ω_r	k_p	k_r	n_s	M_o
0.1 ω_u	0.67	0.23	1.15	0.00
0.2 ω_u	0.65	0.22	1.65	0.00
0.5 ω_u	0.51	0.18	1.77	0.02
2.0 ω_u	-2.04	-0.71	-	-
5.0 ω_u	-16.29	-5.69	-	-

TABLE IV
PERFORMANCE MEASUREMENTS ($T = 0.5$ S) - TL POINT

ω_r	k_p	k_r	n_s	M_o
0.1 ω_u	0.52	0.07	3.36	0.06
0.2 ω_u	0.50	0.06	3.41	0.05
0.5 ω_u	0.39	0.05	2.73	0.00
2.0 ω_u	-1.57	-0.20	-	-
5.0 ω_u	-12.63	-1.65	-	-

TABLE V
PERFORMANCE MEASUREMENTS ($T = 1.0$ S) - ZN POINT

ω_r	k_p	k_r	n_s	M_o
0.1 ω_u	1.00	0.27	0.91	0.00
0.2 ω_u	0.98	0.26	1.21	0.00
0.5 ω_u	0.76	0.20	2.85	0.155
2.0 ω_u	-3.05	-0.82	6.55	0.180
5.0 ω_u	-5.35	-1.43	-	-

TABLE VI
PERFORMANCE MEASUREMENTS ($T = 1.0$ S) - TL POINT

ω_r	k_p	k_r	n_s	M_o
0.1 ω_u	0.78	0.08	2.43	0.06
0.2 ω_u	0.76	0.07	2.93	0.06
0.5 ω_u	0.59	0.06	2.03	0.00
2.0 ω_u	-2.37	-0.23	5.43	0.09
5.0 ω_u	-4.14	-0.41	-	-

TABLE VII
PERFORMANCE MEASUREMENTS ($T = 5.0$ S) - ZN POINT

ω_r	k_p	k_r	n_s	M_o
0.1 ω_u	4.03	0.48	2.34	0.067
0.2 ω_u	3.91	0.47	1.66	0.112
0.5 ω_u	3.05	0.36	8.35	0.389
2.0 ω_u	-12.22	-1.46	3.94	0.081
5.0 ω_u	-32.59	-3.90	2.14	0.00

$$G_2(s) = \frac{1}{(s+1)^n} \tag{24}$$

$$G_3(s) = \frac{1}{(s+1)(1+\alpha s)(1+\alpha^2 s)(1+\alpha^3 s)}. \tag{25}$$

For each class, different values of the model parameters (respectively T, n and α) have been considered, and tests have been made for different frequencies of the sinusoidal reference signal - ω_r. As in the example presented in the Section V, the first step in the test is the relay experiment, for which $d = 2$ and $b = 1$ has been used in all tests. Based on the relay experiment it was possible to determine the amplitude A_u and the period T_u of the self oscillating part of the signal. Only results with the Ziegler-Nichols point are presented.

TABLE VIII
PERFORMANCE MEASUREMENTS ($T = 5.0$ S) - TL POINT

ω_r	k_p	k_r	n_s	M_o
0.1 ω_u	3.12	0.14	1.11	0.11
0.2 ω_u	3.03	0.13	1.72	0.07
0.5 ω_u	2.37	0.10	3.92	0.25
2.0 ω_u	-9.47	-0.42	5.46	0.09
5.0 ω_u	-25.26	-1.12	2.77	0.00

TABLE IX
PERFORMANCE MEASUREMENTS ($T = 0.1$ S)

ω_r	k_p	k_r	n_s	M_o
0.1 ω_u	6.30	13.20	2.68	0.05
0.2 ω_u	6.11	12.80	2.00	0.06
0.5 ω_u	4.77	10.00	5.91	0.33
2.0 ω_u	-19.09	-40.00	74.47	0.10
5.0 ω_u	-152.78	-320.00	11.41	0.00
10.0 ω_u	-630.25	-1320.00	22.33	0.00

TABLE X
PERFORMANCE MEASUREMENTS ($T = 1.0$ S)

ω_r	k_p	k_r	n_s	M_o
0.1 ω_u	1.90	0.57	1.52	0.08
0.2 ω_u	1.84	0.55	0.93	0.04
0.5 ω_u	1.44	0.43	5.27	0.25
2.0 ω_u	-5.76	-1.72	6.33	0.04
5.0 ω_u	-46.12	-13.80	10.95	0.00
10.0 ω_u	-190.26	-56.93	21.80	0.00

For the process (23) with $T = 0.1$ seconds, the amplitude and period of oscillation are $A_u = 0.16$ and $T_u = 0.6$ seconds, resulting in a self oscillation frequency $\omega_u = 10.47$ rad/s. The set of data with the obtained results is synthesized in Table IX. Another set of simulations was realized for this process class considering the parameter value $T = 1.0$ second. In this case, the amplitude for the self oscillating part of the process output signal is $A_u = 0.53$ with a period $T_u = 4.2$ seconds. Table X summarizes the results obtained for this process.

For the process class in equation (24) also two set of experiments were performed, with $n = 3$ and $n = 5$. For the case with $n = 3$ the amplitude and the period of the self oscillating part of the output signal are, respectively, $A_u = 0.33$ and $T_u = 3.7$ seconds, with $\omega_u = 1.69$ rad/s, whereas for $n = 5$ we get $A_u = 0.85$ and $T_u = 8.8$ seconds, with $\omega_u = 0.71$ rad/s. The resulting settings of the PR controller and the performance measures for this process are given in Tables XI and XII.

TABLE XI
PERFORMANCE MEASUREMENTS ($n = 3$)

ω_r	k_p	k_r	n_s	M_o
0.1 ω_u	3.05	1.03	2.04	0.08
0.2 ω_u	2.96	1.00	1.30	0.08
0.5 ω_u	2.31	0.78	8.44	0.36
2.0 ω_u	-9.25	-3.14	6.39	0.06
5.0 ω_u	-74.08	-25.16	10.0	0.00
10.0 ω_u	-305.57	-103.78	19.86	0.00

TABLE XII
PERFORMANCE MEASUREMENTS ($n = 5$)

ω_r	k_p	k_r	n_s	M_o
0.1 ω_u	1.18	0.17	1.05	0.07
0.2 ω_u	1.15	0.16	1.28	0.02
0.5 ω_u	0.90	0.13	2.56	0.14
2.0 ω_u	-3.59	-0.51	5.54	0.00
5.0 ω_u	-28.76	-4.10	-	-
10.0 ω_u	-118.63	-16.94	-	-

TABLE XIII
PERFORMANCE MEASUREMENTS ($\alpha = 0.1$)

ω_r	k_p	k_r	n_s	M_o
0.1 ω_u	40.33	241.37	7.49	0.12
0.2 ω_u	39.11	234.05	32.05	0.27
0.5 ω_u	30.55	182.85	-	-
2.0 ω_u	-122.23	-731.42	8.09	0.29
5.0 ω_u	-977.84	-5851.40	5.11	0.00
10.0 ω_u	-4033.60	-24137.00	8.83	0.00

TABLE XIV
PERFORMANCE MEASUREMENTS ($\alpha = 0.9$)

ω_r	k_p	k_r	n_s	M_o
0.1 ω_u	1.57	0.36	1.33	0.08
0.2 ω_u	1.53	0.35	0.78	0.04
0.5 ω_u	1.19	0.28	3.59	0.23
2.0 ω_u	-4.77	-1.11	5.38	0.00
5.0 ω_u	-38.20	-8.88	34.05	0.09
10.0 ω_u	-157.56	-36.66	217.12	0.31

Finally, the class of processes presented in the transfer function (25) has been considered, with two different values for the parameter α. For $\alpha = 0.1$, the amplitude and period of oscillation associated with the self oscillating part of the output signal are $A_u = 0.025$ and $T_u = 0.21$ seconds, corresponding to $\omega_u = 29.91$ rad/s; $\alpha = 0.9$ results in $A_u = 0.64$ and $T_u = 5.4$ seconds, with $\omega_u = 1.16$ rad/s. All the results concerning the set of tests realized for each process are synthesized in Tables XIII and XIV.

It is seen in the results presented in this Section that:

- for $\omega_r < \omega_u$ the performance is acceptable and similar to what is obtained with the forced oscillation method in PI tuning; stability is observed in all cases except one;
- for $\omega_r < \omega_u$ the performance deteriorates as the reference frequency approaches the ultimate frequency, as expected from the analysis;
- for $\omega_r > \omega_u$ the gains are negative and instability arises in approximately 25% of cases.

VII. CONCLUSIONS

In this paper we have proposed a standard controller structure for Proportional Resonant controllers and a tuning method for this class of controllers. The method is inspired by the forced oscillation method for tuning of PI controllers, and inherits the easiness of its experimental implementation. In order to be able to effectively assess the performance of the closed-loop system, we have also defined performance measures similar to the ones used for constant reference tracking.

When applied to a representative test batch, the tuning rules result in a good performance for almost all cases, for reference frequencies below the ultimate frequency. The resulting performance and robustness are similar to those obtained with the forced oscillation method for PI tuning, in which our method is inspired. For the cases of reference frequencies above the ultimate frequency, instability was observed in a few instances.

The existence of effective and easily understandable tuning rules should contribute to widen the applicability of Internal Model Controllers. We hope that the standardization of the controller structure and of the performance measures may contribute to this effect as well.

It was observed in the test batch that as the reference frequency approaches the ultimate frequency, the closed loop performance is seriously deteriorated. Future work will be concentrated in mitigating this limitation.

REFERENCES

[1] K.J. Åström and T. Hägglund. *PID Controllers, Theory, Design and Tuning*. Instrument Society of America, Research Triangle Park, NC - USA, 1995.

[2] L. Campestrini, L.C. Stevanatto, and A.S. Bazanella. Tuning of multivariable decentralized controllers through the ultimate point method. *IEEE Transactions on Control Systems Technology*, 17(6):1270–1281, 2009.

[3] S. Fukuda and T. Yoda. A novel current-tracking method for active filters based on sinusoidal internal model. *IEEE Transactions on Industry Applications*, 37(3):888–895, 2001.

[4] D. Halim and S.O.R. Moheimani. Spatial resonant control of flexible structures - application to a piezoelectric laminate beam. *IEEE Transactions on Control Systems Technology*, 9(1):37–53, 2001.

[5] W. Luyben, B. Tyreus, and M. Luyben. *Plantwide Process Control*. McGraw-Hill, 2nd edition, 1998.

[6] S.O.R. Moheimani and B.J.G. Vautier. Resonant control of structural vibration using charge-driven piezoelectric actuators. *IEEE Transactions on Control Systems Technology*, 13(6):1021–1035, 2005.

[7] L.F.A. Pereira, J.V. Flores, G. Bonan, D.F. Coutinho, and J.M. Gomes da Silva Jr. Multiple resonant controllers for uninterruptible power supplies - a systematic robust control design approach. *IEEE Transactions on Industrial Electronics*, 61(3):1528–1538, 2014.

[8] D. Rupp and L. Guzzella. Iterative tuning of internal model controllers with application to air/fuel ratio control. *IEEE Transactions on Control Systems Technology*, 18(1):177–184, 2009.

[9] R. Teodorescu, F. Blaajberg, M. Lisserre, and P.C. Loh. Proportional-resonant controllers and filters for grid-connected voltage-source converters. *IEE Proceedings on Electric Power Applications*, 153(5):750–762, 2006.

[10] J.G. Ziegler and N.B. Nichols. Optimum settings for automatic controllers. *Transactions of the ASME*, 64:759–768, 1942.